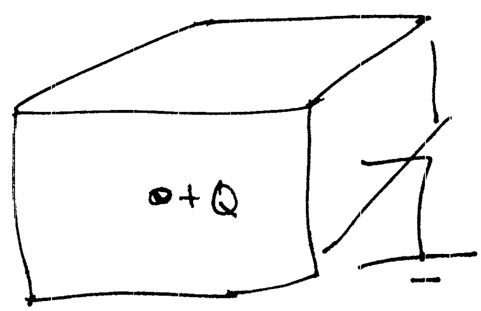
Boundary Value Problems

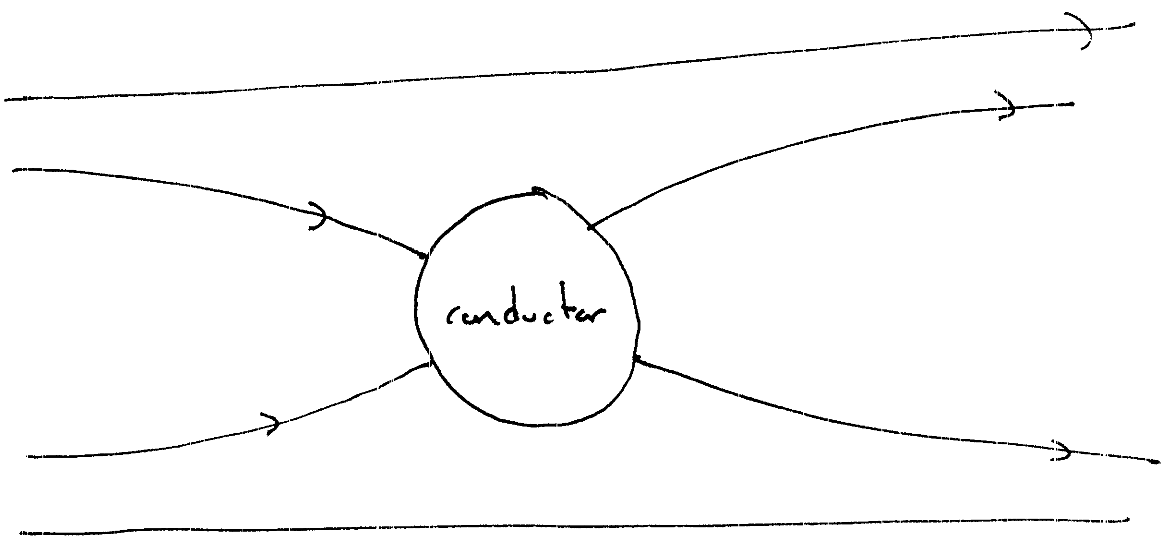
Consider a specification of a physical system where we know the location of the charge in a volume V and something about the potential on the boundary of V.

This would be the case if we put a point charge in a grounded conducting cube,



Since we can build this physically, this must be enough information to uniquely ~~spe~~ determine the field.

We can also place an object in an imposed external field; for example a conductor



In this case, we know the field on the boundary $\propto \left(\frac{\partial V}{\partial n}\right)$ and if we know the charge on the conductor we should be able to find a unique solution, since we can make the system physically.

Note, in this case the volume V is the space outside the conductor.

Uniqueness Thms

When is $\nabla^2 V = -\rho/\varepsilon_0$ uniquely determined?

Dirichlet Problem - The electric potential is uniquely determined in the volume V if

- The location and size of all charge in the volume is fixed
- The potential is fixed on the surface bounding V.

Neumann Problem - The electric potential is uniquely determined in the volume V if

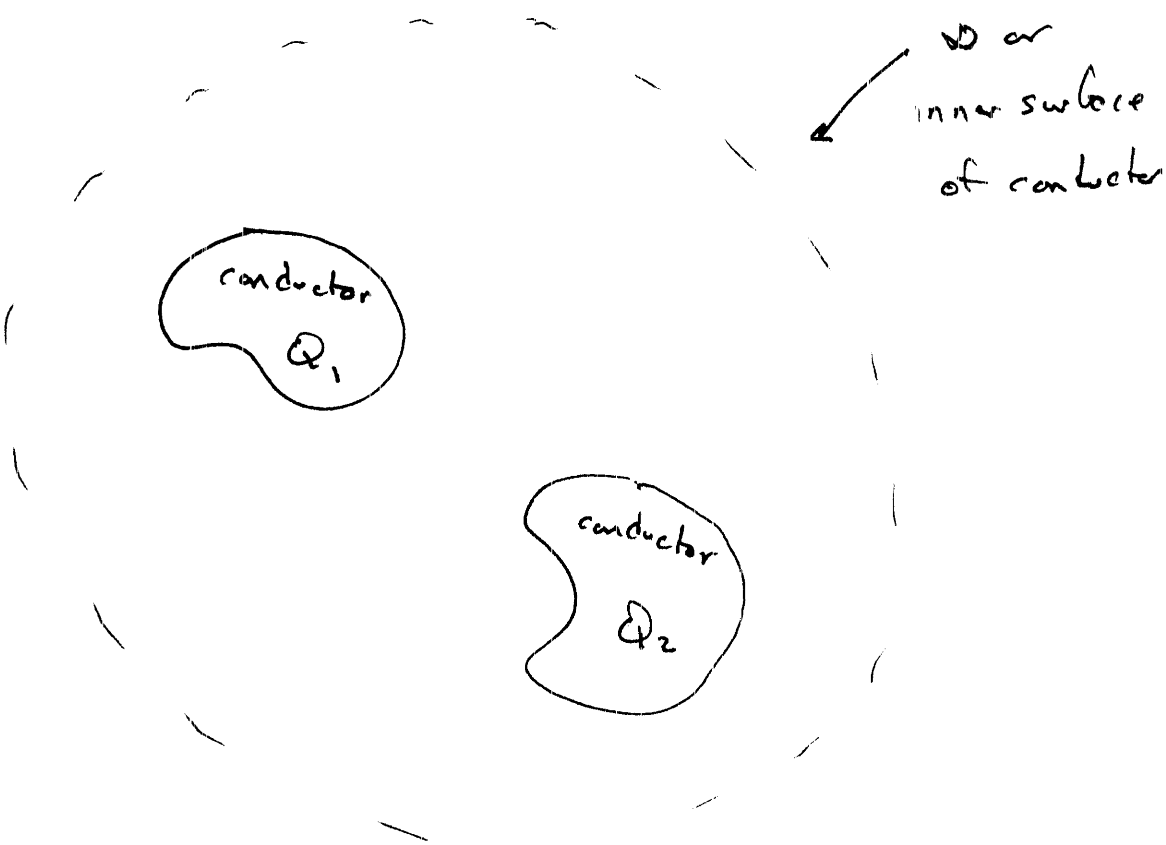
- The location of all charge within V is specified.
- The normal derivative of V, $\frac{\partial V}{\partial n}$, is specified on the surface bounding V.

$\Rightarrow$ Note, if the bounding surface is a conductor, this requires the specification of the surface charge.

⇒ These conditions can be mixed with part of the boundary Dirichlet and part Neumann and a unique solution exists.

Griffiths' Uniqueness Thm II

If a volume V is enclosed by conductors or has a bounding surface at ∞ and the charge density ρ is fixed in the volume, then the field is uniquely determined if the total charge on any conductor in the volume is given.



So what do we have to do?

Solve $\nabla \cdot \vec{E} = \rho/\epsilon_0$

$\nabla \times \vec{E} = 0$

But, if $\vec{E} = -\nabla V$ the curl eqn is automatically taken care of. So the full problem is

$$\nabla^2 V = -\rho/\epsilon_0 \qquad \text{Poisson's Eqn}$$

<u>Dfn Homogeneous</u> – An equation is homogeneous if two solutions f_1 and f_2 imply a third solution $f_1 + f_2$.

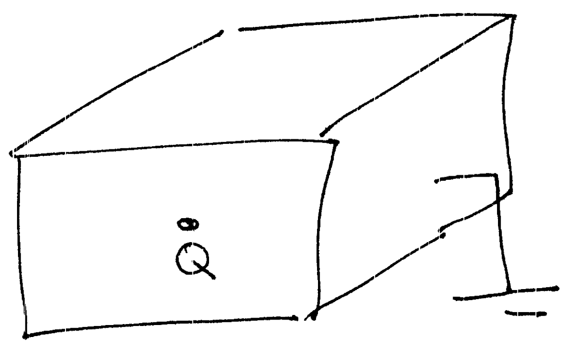
Poisson's equation is not homogeneous but Laplace's eqn $\nabla^2 V = 0$ is.

Our strategy will be to find some solution to Poisson's equation, called the particular solution V_p. This will take care of the ρ in the region under consideration. We will then meet the boundary conditions by taking some combination of the homogeneous solutions to Laplace's equation, V_h.

$$V_h = \sum a_i f_i$$

where a_i is a constant and f_i is a solution to $\nabla^2 f_i = 0$ meeting the boundary conditions.

For example, a point charge in the center of a grounded conducting cube could be solved by



a particular solution $V_p = \frac{kQ}{r}$ to solve

$$\nabla^2 V = -\rho/\varepsilon_0$$

and an infinite series $V_h = \sum a_i f_i(x,y,z)$

where $\nabla^2 f_i = 0$ and a_i are chosen to make $V=0$ on the surface of the cube.

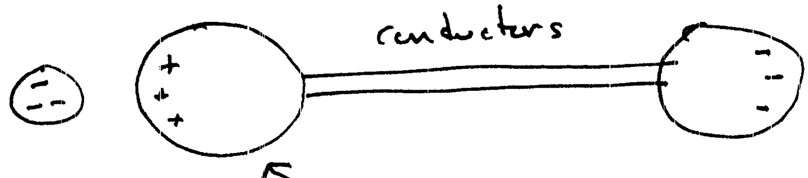
An interesting property of Laplace's eqn.

(1) The potential at the point $\vec{r}$ is the average of the potential over a sphere of any radius R

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V \, da$$

$\Rightarrow$ V can have no local minima or maxima; all extreme values must be on the boundaries.

$\Rightarrow$ Consider



What happens if you ground here?

- Negative charge still goes to ground because that's global minimum of energy
- No local minimum of potential.