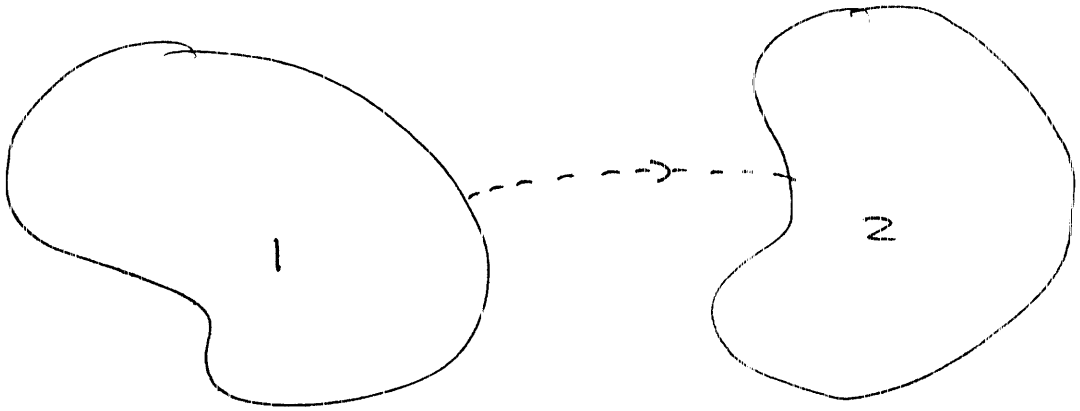
Capacitance

Consider two conductors, initially uncharged,



If the pair are charge by moving +Q from 1 to 2 along the part drawn. The potential difference that develops is proportional to the charge;

$$Q = C|\Delta V|$$

where C is independent of Q, ΔV and only depends on geometry and constants.

Dfn Capacitance

$$C \equiv \frac{Q}{|\Delta V|}$$

- Unit Farad $1F = 1\ C/V$
- $\varepsilon_0 = 8.85\times10^{-12}\ F/m$
- Parallel Plate Capacitor

$$C = \frac{\varepsilon_0 A \leftarrow \text{area}}{d \leftarrow \text{plate separation}}$$

- Isolated Sphere (second plate is sphere at ∞)

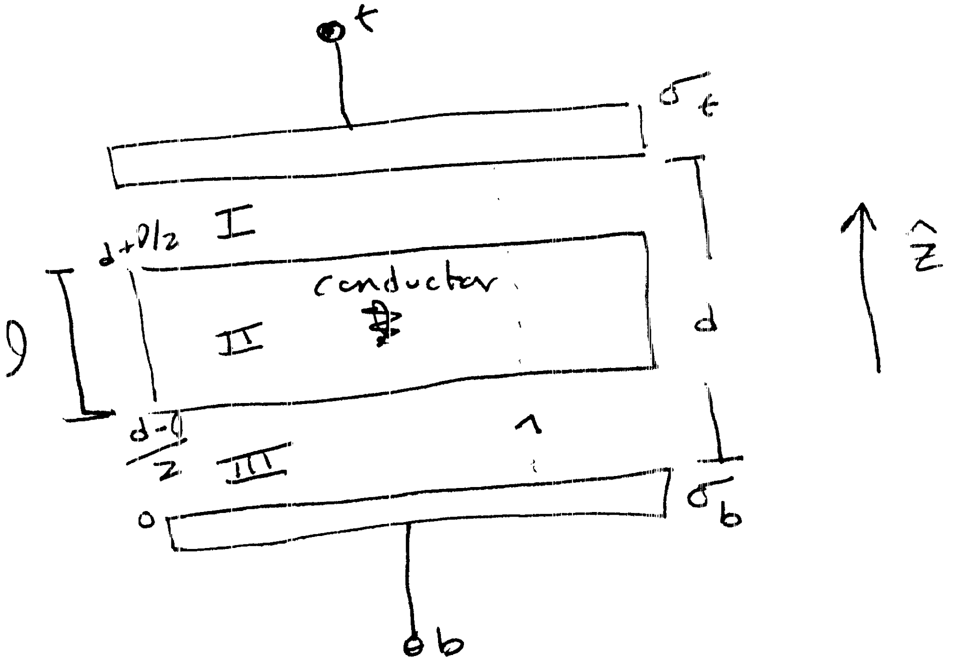
$$C = 4\pi\varepsilon_0 R$$

Energy of a Capacitor

$$U = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q\Delta V$$

Proof $U = W = \int_0^Q \Delta V\, dq = \int_0^Q \frac{q}{C}\, dq = \frac{1}{2}\frac{Q^2}{C}$

Ex Compute capacitance of two parallel plates with separation d with a internal conductor of thickness ℓ



I. Place $+Q$ on bottom plate, $-Q$ on top plate

This generates charge densities

$$\sigma_b = \frac{Q}{A} \equiv \sigma \qquad \sigma_t = \frac{-Q}{A} = -\sigma$$

III <u>Compute Fields</u>

$$\vec{E}_I = \frac{\sigma}{\varepsilon_0}\hat{z} \qquad \vec{E}_{III} = \frac{\sigma}{\varepsilon_0}\hat{z}$$

$$\vec{E}_{II} = 0$$

Compute Potential Difference

$$\Delta V_{ab} = -\int_0^d \vec{E}\cdot d\vec{l} \qquad d\vec{l} = \hat{z}\,dx$$

$$= -\int_0^{\frac{d-l}{2}} \vec{E}_{I}\cdot d\vec{l} - \int_{\frac{d-l}{2}}^{\frac{d+l}{2}} \vec{E}_{II}\cdot d\vec{l} - \int_{\frac{d+l}{2}}^{d} \vec{E}_{III}\cdot d\vec{l}$$

$$= -\int_0^{\frac{d-l}{2}} \frac{\sigma}{\varepsilon_0}\,dz - 0 - \int_{\frac{d+l}{2}}^{d} \sigma/\varepsilon_0\,dx$$

$$= -\left(\frac{d-l}{2}\right)\frac{\sigma}{\varepsilon_0} - \left(d - \frac{d+l}{2}\right)\frac{\sigma}{\varepsilon_0}$$

$$= -(d-l)\frac{\sigma}{\varepsilon_0}$$

sign is correctly negative.

Dfn Capacitance

$$C = \frac{Q}{|\Delta V|} = \frac{\sigma A}{|\Delta V|} = \frac{\sigma A}{(d-l)\sigma/\varepsilon_0}$$

$$= \frac{A\varepsilon_0}{d-l}$$

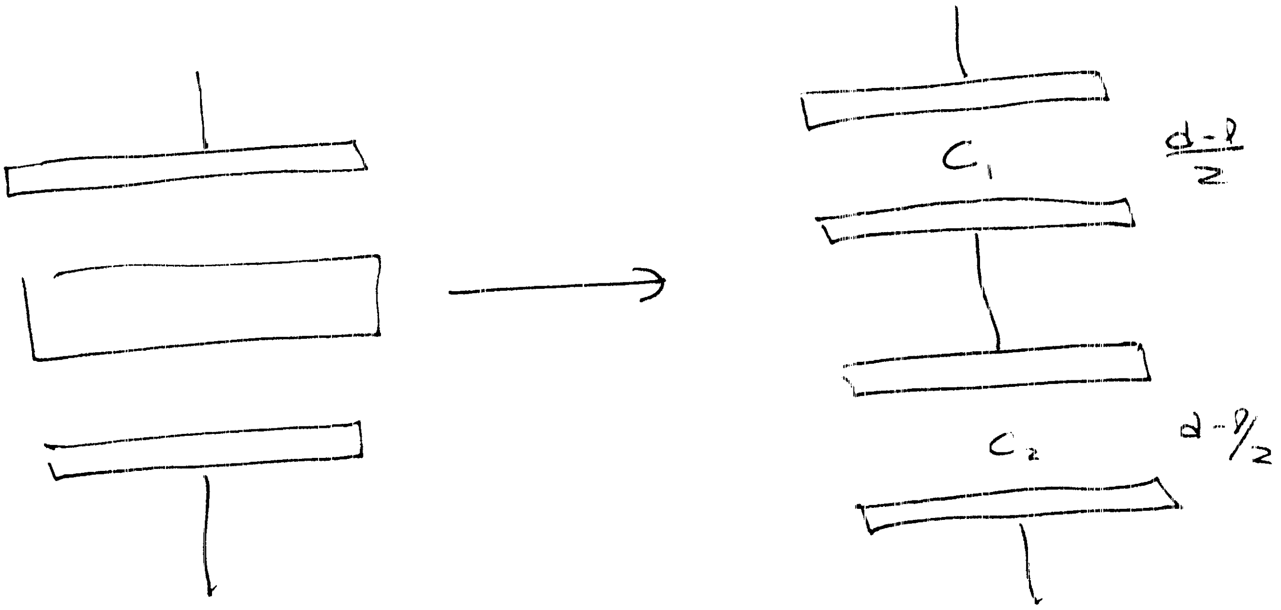
Another way

Capacitors in series add as

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Capacitors add in parallel as

$$C_{eq} = C_1 + C_2$$

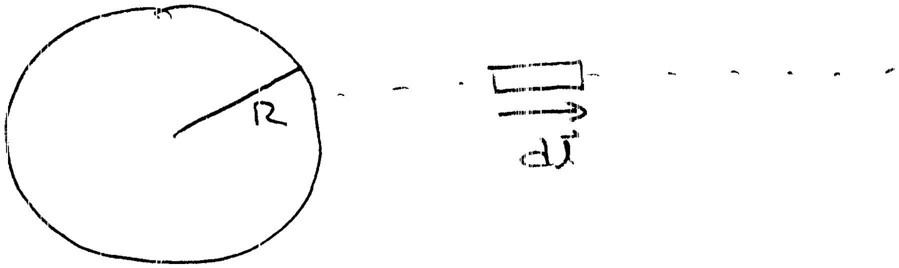


$$C_1 = \frac{\varepsilon A}{d - \ell/2} = C_2 = \frac{2\varepsilon A}{d - \ell}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d-\ell}{2\varepsilon A} + \frac{d-\ell}{2\varepsilon A}$$

$$C_{eq} = \frac{\varepsilon_0 A}{d - l}$$ ✓

Ex Capacitance of isolated sphere of radius R



Add +Q to Sphere

Field $r > R$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Potential Difference - Alway integrate in positive direction

$$d\vec{l} = \hat{r}\,dr$$

$$\Delta V_{R\infty} = -\int_R^\infty \vec{E}\cdot d\vec{l} = -\int_R^\infty \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}\cdot\hat{r}\,dr$$

$$\Delta V_{R\infty} = \frac{Q}{4\pi\varepsilon_0} \left. \frac{1}{r} \right|_R^\infty$$

$$= -\frac{Q}{4\pi\varepsilon_0 R}$$ (correctly negative)

Dfn Capacitance

$$C = \frac{Q}{|\Delta V|} = 4\pi\varepsilon_0 R$$

ex Energy Stored in a thin spherical shell of charge with charge density σ and radius R.

Method I Distribution meets condition for capacitance.

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R} = \frac{Q^2}{8\pi\varepsilon_0 R}$$

$$Q = 4\pi R^2 \cancel{\varepsilon_0} \sigma$$

$$U = \frac{1}{2}\frac{(4\pi)^2 R^4 \sigma^2}{4\pi\varepsilon_0 R} = \frac{2\pi R^3 \sigma^2}{\varepsilon_0}$$

Method II

$$U = \frac{1}{2}\int \rho V \, d\tau =$$

$$= \frac{1}{2}\int \sigma V \, da \qquad V = \frac{Q}{4\pi\varepsilon_0 R}$$

$$= \frac{\sigma Q}{2\cdot 4\pi\varepsilon_0 R}\oint da = \frac{4\pi R^2 Q \sigma}{2\cdot 4\pi\varepsilon_0 R}$$

$$= \frac{Q^2}{8\pi\varepsilon_0 R}$$

Method III

Energy density $= \frac{1}{2}\varepsilon_0 E^2$

$$= \frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$$

Total Energy

$$U = \int_{\text{space}} \frac{1}{2}\varepsilon_0 E^2 \, d\tau$$

Since spherically symmetric

$$d\tau = 4\pi r^2 dr$$

$$U = \int_R^\infty \frac{Q^2}{32\pi^2\varepsilon_0 r^4} \cdot 4\pi r^2 dr$$

$$= \frac{Q^2}{8\pi\varepsilon_0} \int_R^\infty \frac{1}{r^2}$$

$$= \frac{Q^2}{8\pi\varepsilon_0} \left(-\frac{1}{r} \right)_R^\infty = \frac{Q^2}{8\pi\varepsilon_0 R}$$