Complex Numbers

A complex number is a combination of two real numbers a, b and i.

$$z = a + bi$$

where $i^2 = -1$.

Real Part

$Re(z) = a$

Imaginary Part

$Im(z) = b$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Complex numbers make many calculations much easier.

Ex Find double angle formulas $\sin 2\theta$, $\cos 2\theta$

$$e^{i2\theta} = e^{i\theta} e^{i\theta}$$

$$= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$

$$= (\cos^2\theta + i^2 \sin\theta) + 2i\sin\theta\cos\theta$$

$$= \cos 2\theta + i\sin 2\theta \quad \text{Euler's formula}$$

For two complex numbers to be equal, the real and the imaginary parts must be equal.

$$z_1 = a_1 + i b_1 \qquad z_2 = a_2 + i b_2$$

If $z_1 = z_2$,

$$Re(z_1) = a_1 = Re(z_2) = a_2$$

$$Im(z_1) = b_1 = Im(z_2) = b_2$$

Since the real and imaginary parts of a complex number are independent, if we can solve an equation in terms of a complex $\tilde{f}$

$$\frac{\partial^2 \tilde{f}}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \tilde{f}}{\partial t^2} = 0$$

Then the physical (real) solution is

$$f = Re(\tilde{f})$$

~~Modulus (z)~~

Complex conjugate of $z \equiv z^*$

If $z = a + ib$, $z^* = a - ib$

Modulus of z

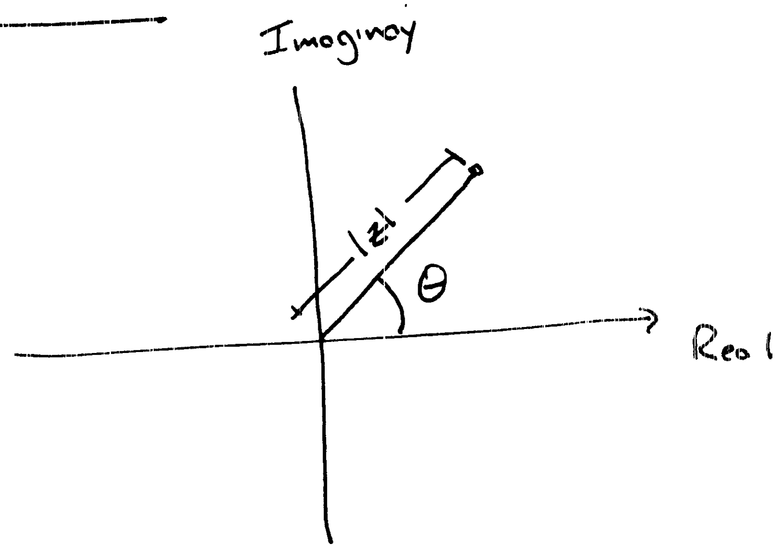
$$|z| = \sqrt{z^* z} = \sqrt{a^2+b^2}$$

Any complex number can be written as

$$z = |z| e^{i\theta}$$

Note
$$z^* z = |z|^* e^{-i\theta} |z| e^{i\theta}$$
$$= |z|^2$$

Complex Plane



Note
$$+i = e^{i\pi/2}$$
$$-1 = e^{i\pi}$$

Roots of complex numbers

What is $\sqrt{i}$?

$$i = e^{+i\pi/2}$$

$$\sqrt{i} = \sqrt{e^{+i\pi/2}} = e^{i\pi/4}$$

Complex Waves

$$f(x,t) = A\cos(kx - \omega t + \delta)$$

$$= \mathrm{Re}\left(\tilde{f}(x,t)\right)$$

$$\tilde{f}(x,t) = \tilde{A}\, e^{i(kx-\omega t)}$$

$$\tilde{A} = A\, e^{i\delta}$$

Wave Packets

Arbitrary profiles $g(x)$, $h(x)$ can be built up out of sines and cosines

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)\, e^{i(kx-\omega t)}\, dk$$

Note, this is the same are using solutions to Laplace's equation to build up arbitrary functions, but there was nothing to quantize the solutions.