Conservation

Consider some of the conservation laws we work with in physics;

Conservation of Mass (ρ mass density)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Conservation of Charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

Conservation of Energy

$$\sum U_{before} = \sum U_{after}$$

Conservation of Momentum

$$\sum_{before} \vec{P} = \sum_{after} \vec{P}$$

Note how much weaker the statements of conservation of energy and momentum are. For charge and mass, we can see how the quantity is conserved. If we seprate the universe into a system and the environment, a change in the charge of the system results from a flow of charge through the boundry of the system.

$$\int_V \frac{\partial \rho}{\partial t}\, d\tau + \int_V \nabla \cdot \vec{J}\, d\tau = 0$$

$$\frac{d}{dt} \underbrace{\int_V \rho\, d\tau}_{\text{total charge}} + \underbrace{\int_S \vec{J} \cdot d\vec{a}}_{\text{charge flowing out of system.}} = 0$$

For momentum and energy, we have no idea how conservation is accomplished. For all we know ~~energy~~ energy pixies teleport energy between the system and the environment

We will rectify this situation by writing a continuity equation for electromagnetic energy.

Consider a system of charged particles isolated from the environment. The moving charge creates electric and magnetic fields. The fields do work on the charges changing their kinetic and potential energy.

The energy of the system can be separated into the energy of mechanical energy U_{mech} and the energy of the electromagnetic fields U_{em}. Energy is transferred from the fields to mechanical energy as the fields do work on the particles -

$$\frac{dU_{mech}}{dt} = \frac{d}{dt}\int_V \vec{F}_{em} \cdot d\vec{l} = \int \vec{F}_{em} \cdot \vec{v}\, d\tau$$

$$\vec{F}_{em} = q\vec{E} + q\vec{v} \times \vec{B}$$

and the force density

$$\vec{f}_{em} = \rho\vec{E} + \rho\vec{v} \times \vec{B}$$

Naturally, $\vec{v} \cdot (\vec{v} \times \vec{B}) = 0$

$$\frac{dU_{mech}}{dt} = \int_V \rho \vec{E} \cdot \vec{v} \, d\tau$$

$$= \int_V \vec{E} \cdot \vec{J} \, d\tau$$

which is simply our Joule heating term, where now the energy lost to the field is not required to appear as heat.

Eliminate $\vec{J}$ using Ampere's Law

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \vec{E} \cdot (\nabla \times \vec{B}) - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

<u>Second Term</u>

$$\frac{\partial \vec{E} \cdot \vec{E}}{\partial t} = \frac{\partial E^2}{\partial t} = 2 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial E^2}{\partial t}$$

First term - Vector identity (6)

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\vec{B} \Rightarrow \vec{E} \qquad \vec{A} \Rightarrow \vec{B}$$

$$\nabla \cdot (\vec{B} \times \vec{E}) = \vec{E} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{E})$$

Faraday's Law $\quad \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

$$\nabla \cdot (\vec{B} \times \vec{E}) = \vec{E} \cdot (\nabla \times \vec{B}) + \vec{B} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$-\nabla \cdot (\vec{E} \times \vec{B}) = \vec{E} \cdot (\nabla \times \vec{B}) + \frac{1}{2} \frac{\partial B^2}{\partial t}$$

$$\vec{E} \cdot (\nabla \times \vec{B}) = -\nabla \cdot (\vec{E} \times \vec{B}) - \frac{1}{2} \frac{\partial B^2}{\partial t}$$

$$\vec{E} \cdot \vec{J} = -\frac{1}{\mu_0} \nabla \cdot (\vec{E} \times \vec{B}) - \frac{1}{2\mu_0} \frac{\partial B^2}{\partial t} - \frac{1}{2} \epsilon_0 \frac{\partial E^2}{\partial t^2}$$

Putting it all together

$$\frac{dU_{mech}}{dt} = -\frac{d}{dt}\int_V \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right) d\tau$$

$$-\frac{1}{\mu_0}\int_V \nabla \cdot (\vec{E}\times\vec{B})\, d\tau$$

Electromagnetic Energy Density

$$u_{em} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

$\Rightarrow$ Had pieces before, this shows we do not pick up additional stuff from interaction between $\vec{E} + \vec{B}$ or time dependent pieces.

~~Poynting~~ **Poynting Vector** - Energy flux per unit area,

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$|\vec{S}| = \frac{\text{Energy}}{\text{Area}\cdot\text{Time}} = \frac{\text{Power}}{\text{Area}}$$

Poynting's Thm

Divergence Thm

$$\frac{dU_{mech}}{dt} = -\frac{d}{dt}\int_V u_{em}\, d\tau - \oint_S \vec{S}\cdot d\vec{a}$$

If we express the mechanical energy as a density

$$U_{mech} = \int_V u_{mech}\, d\tau$$

we get

$$\frac{d}{dt}\int_V (u_{mech} + u_{em})\, d\tau = -\oint_S \vec{S}\cdot d\vec{a}$$

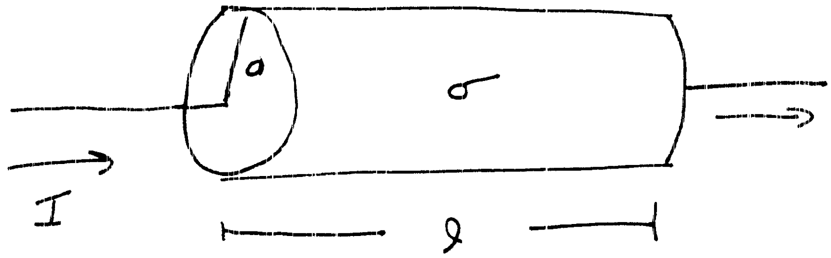
$$= -\int_V (\nabla\cdot\vec{S})\, d\tau$$

or

$$\frac{\partial(u_{mech} + u_{em})}{\partial t} = -\nabla\cdot\vec{S}$$

Continuity equation for energy

Ex



If a current I, flows uniformly in a resistor of radius a, a current density of

$$J = \frac{I}{\pi a^2}$$

results.

By Ohm's Law, this requires an electric field of

$$E = \frac{J}{\sigma} = \frac{I}{\pi a^2 \sigma}$$

Since the field is uniform, the potential difference across the resistor is

$$\Delta V = El = \frac{Il}{\pi a^2 \sigma} = IR$$

The energy converted from EM to mechanical (heat) per unit time is $P = I\Delta V = \dfrac{I^2 l}{\pi a^2 \sigma}$

We could get this directly from Joule heating

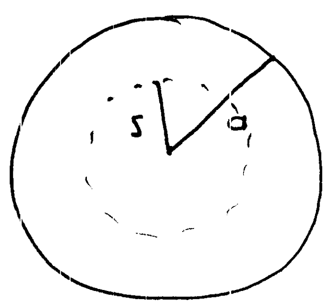
$$P = \int E \cdot J \, d\tau = \frac{1}{\sigma} \int J^2 d\tau$$

$$= \frac{1}{\sigma} \left(\frac{I}{\pi a^2} \right)^2 \pi a^2 l$$

$$= \frac{I^2}{\sigma \pi a^2} l$$

or from Poynting thm

Magnetic Field



z out of page

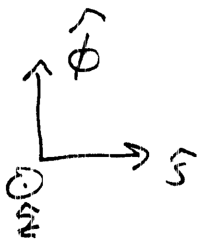
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \pi s^2 J = \mu_0 I \frac{\pi s^2}{\pi a^2}$$

$$= 2\pi s B$$

$$\vec{B} = \frac{\mu_0 I s}{2\pi a^2} \hat{\phi} \longleftarrow \text{RHR}$$

$$\vec{E} = \frac{I}{\pi a^2 \sigma} \hat{z}$$

$\vec{E} \perp \vec{B}$ $\qquad\qquad$ $\hat{s} \times \hat{\phi} = \hat{z}$



$\hat{z} \times \hat{\phi} = -\hat{s}$

Poynting Vector Points Inward

$$\underbrace{\frac{dU_{mech}}{dt}}_{\substack{\shortparallel \\ \text{Increase in} \\ \text{thermal energy.}}} = \underbrace{-\frac{d}{dt}\int u_{em}\, d\tau}_{0 \text{ fields constant}} \underbrace{- \int \vec{S}\cdot d\vec{a}}_{\text{Energy flux}}$$

Poynting Vector at Surface

evaluate B at surface

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\left(\frac{I}{\pi a^2 \sigma}\right)\left(\frac{\mu_0 I a}{2a^2\pi}\right)(-\hat{s})$$

$$= \frac{I^2}{2\pi a^3 \sigma}(-\hat{s})$$

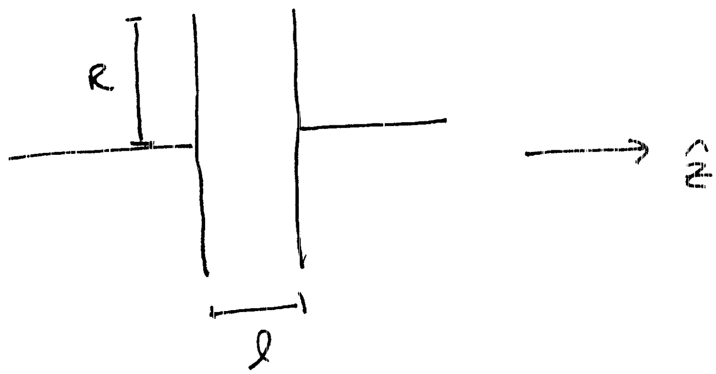
$$\int \vec{S}\cdot d\vec{a} = -2\pi a \ell |\vec{S}| = -\frac{I^2 \ell}{\pi a^2 \sigma} = I\Delta V$$

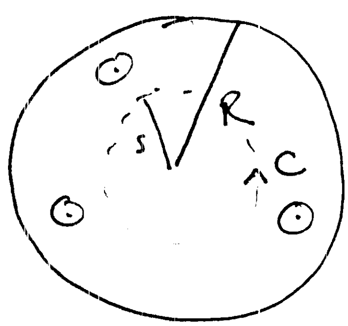
again.

Ex We previously considered a capacitor discharging such that the electric field between the circular plates had time dependence

$$\vec{E} = E_0 e^{-t/\tau} \hat{z}$$

where the plates had radius R. Let the spacing between the plates be ℓ





The displacement current through the surface bounded by C is

$$I_d = \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a}$$

Electric Flux

$$\Phi_e = EA = E_0 \pi s e^{-t/\tau}$$

$\Rightarrow$ Note out of page positive normal.

Displacement Current

$$I_d = \varepsilon_0 \frac{d\Phi_e}{dt} = \frac{-\varepsilon_0 E_0 \pi s^2}{\tau} e^{-t/\tau}$$

Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (I_{enc} + I_d) = \mu_0 I_d$$

by symmetry

$$\oint \vec{B} \cdot d\vec{l} = 2\pi s B = \mu_0 I_d \cancel{\frac{E_0 s}{2\tau}} e^{-t/\tau}$$

The - sign indicates the field is opposite the chosen direction for C.

$$\vec{B} = -\frac{\mu_0 \varepsilon_0 E_0 s}{2\tau} \hat{\phi} e^{-t/\tau}$$

The Poynting Vector is

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$$

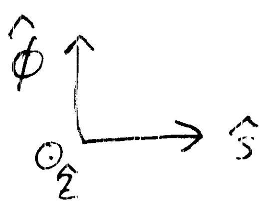
$$= \frac{1}{\mu_0}\left(E_0 e^{-t/\tau}\,\hat{z}\right) \times \left(-\frac{\mu_0 \varepsilon_0 E_0 R}{2\tau} e^{-t/\tau}\,\hat{\phi}\right)$$

where the magnetic field is evaluated at the edge of the capacitor.

$$\vec{S} = -\frac{\varepsilon_0 E_0^2 R}{2\tau} e^{-2t/\tau}\;\hat{z} \times \hat{\phi}$$

Right-handed coordinate system

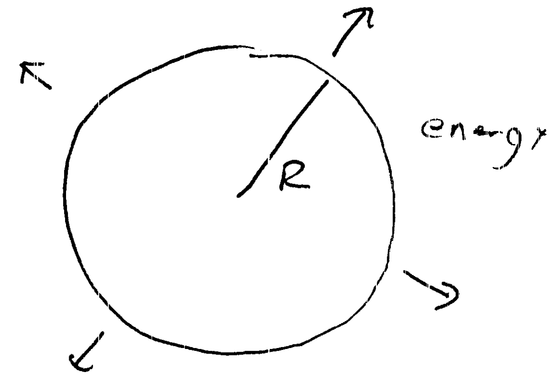
$$\hat{s} \times \hat{\phi} = \hat{z}$$



$$\hat{z} \times \hat{\phi} = -\hat{s}$$

$$\vec{S} = \frac{\varepsilon_0 E_0^2 R}{2\tau} e^{-2t/\tau}\;\hat{s}$$

There is a flux of energy out of the edges of the capacitor.



The total energy flowing of of the capacitor per unit time is

$$P = \frac{\text{Energy}}{\text{time}} = SA = S \cdot 2\pi R \ell$$

$$= \frac{\varepsilon_0 E_0^2 R}{2\tau} e^{-2t/\tau} \cdot 2\pi R \ell = \frac{\varepsilon_0 E_0^2 \pi R^2 \ell}{\tau} e^{-2t/\tau}$$

We can check this against the energy stored in the capacitor.

$$U = \frac{1}{2} C \Delta V^2$$

$$\frac{dU}{dt} = C \Delta V \frac{d\Delta V}{dt}$$

$$C = \frac{\varepsilon_0 A}{\ell} = \frac{\varepsilon_0 \pi R^2}{\ell}$$

$\Delta V = E \ell$ since field uniform

$$\frac{d\Delta V}{dt} = -\frac{E_0 \ell}{\tau} e^{-t/\tau}$$

$$\frac{dU}{dt} = C \Delta V \frac{d\Delta V}{dt} = \left(\frac{\varepsilon_0 \pi R^2}{\ell}\right)\left(E_0 \ell e^{-t/\tau}\right)\left(\frac{-E_0 \ell}{\tau} e^{-t/\tau}\right)$$

$$= -\frac{\varepsilon_0 \pi R^2 E_0^2 \ell}{\tau} e^{-2t/\tau} \quad ✓$$

We could also check this using the densities of the electric and magnetic field.