Momentum

It is fairly easy to find situations where Newton's Third Law appears not to hold in magnetism.

Newton's Third Law is just a statement of the pairwise conservation of momentum

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\frac{d\vec{p}_{12}}{dt} + \frac{d\vec{p}_{21}}{dt} = 0$$

The electromagnetic field must conserve momentum or the universe is in big trouble. If the momentum doesn't reside with the particles it must reside in the fields.

We would like to write a continuity equation for momentum. The change in momentum of an object would then be the total momentum flowing through the surface of the object.

We will use T for the momentum flux -

$$T = \frac{\text{momentum}}{\text{area} \cdot \text{time}}$$

$$\frac{dp}{dt} = \int_S T\, da = \text{Force}$$

Therefore the force on a object is the integral of the momentum flux over the surface.

Now comes the problem, momentum is a vector quantity so we need a x momentum flux, a y momentum flux, and a z momentum flux.

Flux itself is a vector quantity in that it has direction, so we actually need the x momentum crossing an area $dA\,\hat{x}$ per unit time.

So our flux must have two components

$$T_{xx} = \frac{x\text{-momentum}}{x\text{-Area} \cdot \text{time}}$$

$$T_{yx} = \frac{y\text{-momentum}}{x \text{ Area} \cdot \text{time}}$$

To get the momentum per time flowing through a surface, multiple T by the area.

So if T is constant, to get the z-momentum flowing through the surface $A_y \hat{y}$ in the x-z plane we would calculate

$$F_z = \frac{dp_z}{dt} = T_{zy} A_y$$

or if we wanted the other components

$$F_x = \frac{dp_x}{dt} = T_{xy} A_y$$

$$F_y = \frac{dp_y}{dt} = T_{yy} A_y$$

The notation suggests matrix multiplication

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \frac{d\vec{p}}{dt} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ A_y \\ 0 \end{pmatrix}$$

This would allow us to write the momentum per unit time flowing through an arbitrary area as

$$\vec{F} = \frac{d\vec{p}}{dt} = \underbrace{\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}}_{\equiv \overleftrightarrow{T}} \underbrace{\begin{pmatrix} An_x \\ An_y \\ An_z \end{pmatrix}}_{= A\hat{n} = \vec{A}}$$

Stress Tensor ($\overleftrightarrow{T}$)

$$\vec{F} = \frac{d\vec{p}}{dt} = \overleftrightarrow{T} \cdot \vec{A}$$

Note, the dot product yields a vector

While we're working on notation, let

$\vec{T}_x$ be the flux of

x-momentum.

$$\vec{T}_x = (T_{xx}, T_{xy}, T_{xz})$$

If all is well, we will be able to write the divergence this as the time rate of change of the x-momentum density ρ_x

$$\frac{\partial \rho_x}{\partial t} + \nabla \cdot \vec{T}_x = 0$$

and

$$\frac{\partial \rho_y}{\partial t} + \nabla \cdot \vec{T}_y = 0$$

$$\frac{\partial \rho_z}{\partial t} + \nabla \cdot \vec{T}_z = 0$$

Note by convention we will actually have a negative sign

Or more compactly,

$$\frac{\partial \vec{\rho}}{\partial t} + \nabla \cdot \overleftrightarrow{T} = 0$$

$$\nabla \cdot \overleftrightarrow{T} = \left(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} , \frac{\partial}{\partial z} \right) \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Ok, let's figure out what $\overleftrightarrow{T}$ is for the electromagnetic field.

Let the total momentum of an isolated system be separated into the mechanical momentum of the particles and any momentum associated with the fields

$$\vec{P} = \vec{P}_{mech} + \vec{P}_{em}$$

$\vec{P}_{mech}$ changes because the fields exerts forces on the charges. Let ρ be the charge density of the system. The force per unit volume acting on the charges is then

$$\vec{f} = \rho \left(\vec{E} + \vec{v} \times \vec{B} \right) = \rho \vec{E} + \vec{J} \times \vec{B}$$

The total force and therefore the time rate of change of the mechanical momentum is

$$\vec{F} = \int \vec{f}\, d\tau = \frac{d\vec{P}_{mech}}{dt} = \int_V \left(\rho\vec{E} + \vec{J}\times\vec{B}\right) d\tau$$

As before, eliminate ρ, $\vec{J}$ using Maxwell's eqns

$$\rho = \varepsilon_0 \nabla\cdot\vec{E}$$

$$\vec{J} = \frac{\nabla\times\vec{B}}{\mu_0} - \varepsilon_0 \frac{\partial\vec{E}}{\partial t}$$

$$\rho\vec{E} + \vec{J}\times\vec{B} = \varepsilon_0 \vec{E}(\nabla\cdot\vec{E}) + \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} - \varepsilon_0 \frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

Work on this term by term

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

<u>Faraday's Law</u> $\quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} - \vec{E}\times(\nabla\times\vec{E})$$

$$\frac{\partial E}{\partial t} \times \vec{B} = \vec{E} \times (\nabla \times \vec{E}) + \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

So,

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B} = \varepsilon_0 (\nabla \cdot \vec{E})\vec{E} - \frac{1}{\mu_0} \vec{B} \times (\nabla \times \vec{B})$$

$$- \varepsilon_0 \vec{E} \times (\nabla \times \vec{E}) - \varepsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

where I used $(\nabla \times \vec{B}) \times \vec{B} = - \vec{B} \times (\nabla \times \vec{B})$

~~Let's increase the~~

Vector identity –

$$\nabla (\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A})$$
$$+ (A \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A}$$

Let $\vec{A} \rightarrow \vec{E}$, $\vec{B} \rightarrow \vec{E}$

$$\nabla E^2 = \vec{E} \times (\nabla \times \vec{E}) + \vec{E} \times (\nabla \times \vec{E}) + (\vec{E} \cdot \nabla) \vec{E}$$
$$+ (\vec{E} \cdot \nabla) \vec{E}$$

$$= 2 \vec{E} \times (\nabla \times \vec{E}) + 2 (\vec{E} \cdot \nabla) \vec{E}$$

$$\vec{E} \times (\nabla \times \vec{E}) = \frac{1}{2} \nabla E^2 - (\vec{E} \cdot \nabla) \vec{E}$$

$$\vec{f} = \varepsilon_0 (\nabla \cdot \vec{E}) \vec{E} - \frac{1}{2} \varepsilon_0 \nabla E^2 + \varepsilon_0 (\vec{E} \cdot \nabla) \vec{E}$$

$$- \frac{1}{2\mu_0} \nabla B^2 + \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B}$$

$$- \varepsilon_0 \frac{\partial}{\partial t} \left(\vec{E} \times \vec{B} \right)$$

Collect stuff that sort looks the same and using $\nabla \cdot \vec{B} = 0$

$$\vec{f} = \varepsilon_0 \left((\nabla \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \nabla) \vec{E} \right)$$

$$+ \frac{1}{\mu_0} \left((\nabla \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \nabla) \vec{B} \right)$$

$$- \frac{1}{2} \nabla \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)$$

$$- \varepsilon_0 \frac{\partial}{\partial t} \left(\vec{E} \times \vec{B} \right)$$

The third term is the gradient of the energy density and the fourth term is related to the time rate of change of $\vec{S}$.

The first ~~two~~ three terms are $\overleftrightarrow{T}$

Maxwell Stress Tensor ($\overleftrightarrow{T}$)

$$T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right)$$
$$+ \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

For example,

$$T_{xx} = \varepsilon_0 \left(E_x^2 - \frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left(B_x^2 - \frac{1}{2} B^2 \right)$$

$$= \varepsilon_0 \left(\frac{1}{2} E_x^2 - \frac{1}{2} E_y^2 - \frac{1}{2} E_z^2 \right)$$
$$+ \frac{1}{\mu_0} \left(\frac{1}{2} B_x^2 - \frac{1}{2} B_y^2 - \frac{1}{2} B_z^2 \right)$$

$$T_{xy} = \varepsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y$$

In terms of $\overleftrightarrow{T}$

$$\vec{f} = \nabla \cdot \overleftrightarrow{T} - \varepsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

The total force on the volume V is

$$\vec{F} = \int_V \vec{f}\, d\tau = \int \nabla \cdot \overleftrightarrow{T}\, d\tau - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S}\, d\tau$$

$$= \frac{d\vec{p}_{mech}}{dt} = -\frac{d\vec{p}_{em}}{dt} - \oint \overleftrightarrow{\wp} \cdot d\vec{a}$$

momentum flux density

Comparing terms – The momentum flux density out of the surface S is $-\overleftrightarrow{T}$.

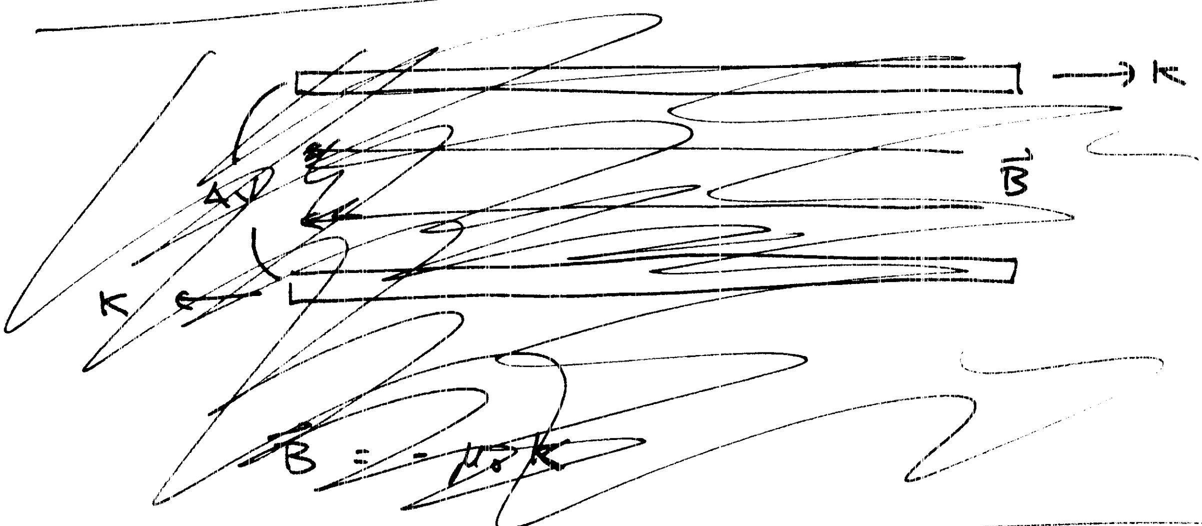
The momentum density $\vec{\wp}_{em}$ associated with the fields is

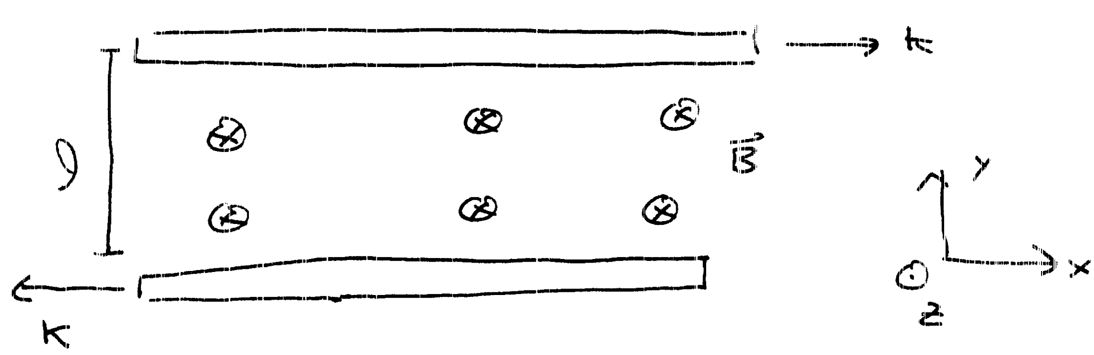
$$\vec{\wp}_{em} = \varepsilon_0 \mu_0 \vec{S}$$

The continuity eqn for momentum is

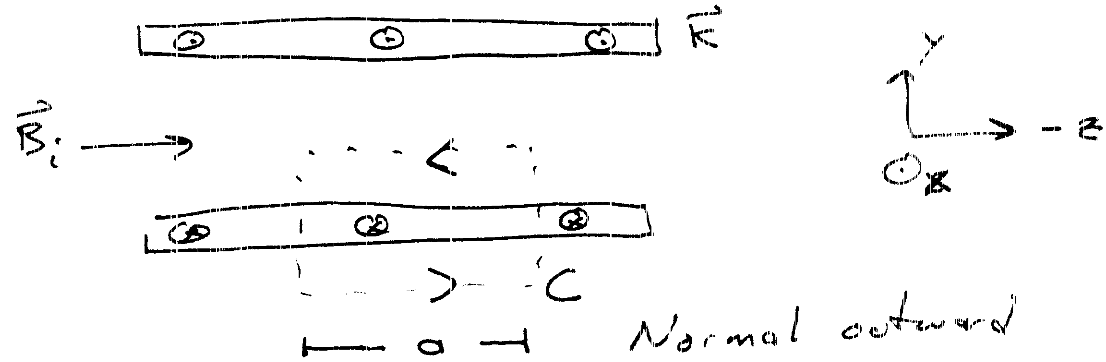
$$\frac{d}{dt}\left(\vec{\wp}_{mech} + \vec{\wp}_{em}\right) = \nabla \cdot \overleftrightarrow{T}$$

<u>Ex</u> Compute momentum per unit area between two current sheets carrying current K held at potential difference ΔV





End View



$$I_{enc} = -aK$$

$$\oint \vec{B} \cdot d\vec{l} = -B_i a = \mu_0 I_{enc} = -a K \mu_0$$

$$\vec{B}_i = -\mu_0 K \hat{z}$$

The electric field is $\vec{E} = \frac{V}{\ell} \hat{y}$

The Poynting vector is then

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = -\frac{VK}{\ell} \hat{y} \times \hat{z}$$

$$\vec{S} = -\frac{VK}{\ell} \hat{x}$$

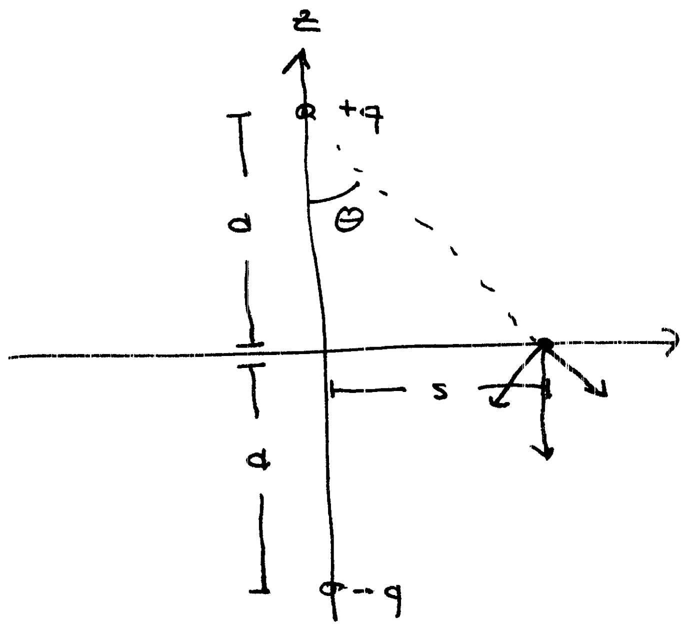
The momentum density $\vec{\wp}_{em} = -\mu_0 \varepsilon_0 \frac{VK}{\ell} \hat{x}$

The momentum per unit area is

$$\vec{P} = \vartheta \vec{\sigma}_{em} = -\mu_0 \varepsilon_0 V K \hat{x}$$

Note, this is crazy since the system is at rest. This momentum is balanced by a momentum associated with the current that is due to relativity. Note, for normal current speeds relativity could be important because $\mu_0 \varepsilon_0$ is so small.

8.4(b)



$$\vec{E} = -\frac{2kq}{(a^2+s^2)} \cdot \underbrace{\frac{a}{\sqrt{a^2+s^2}}}_{\cos\theta} \hat{z}$$

$$\vec{E} = \frac{-qa\,\hat{z}}{2\pi\epsilon_0 (a^2+s^2)^{3/2}}$$

The force on the $+q$ charge is found by integrating $\overleftrightarrow{T}$ over the bounding surface with outward surface normal $\hat{n} = -\hat{z}$.

The force is obviously in the $-\hat{z}$ direction

$$\vec{F} = \underbrace{-\hat{z}}_{\text{from normal}} \int_{\text{plane}} T_{zz}\, da$$

$$T_{zz} = \epsilon_0 \left(E_z^2 - \frac{1}{2}E^2 \right) + \frac{1}{\mu_0}\underbrace{\left(B_z^2 - \frac{1}{2}B^2 \right)}_{=0}$$

$E^2 = E_z^2$ in this case

$$T_{zz} = \frac{1}{2}\epsilon_0 E_z^2 = \frac{q^2 a^2}{8\pi^2 \epsilon_0 (a^2+s^2)^3}$$

$$\vec{F}_z = -\hat{z}\int_{\text{plane}} T_{zz}\, ds\, s\, d\phi$$

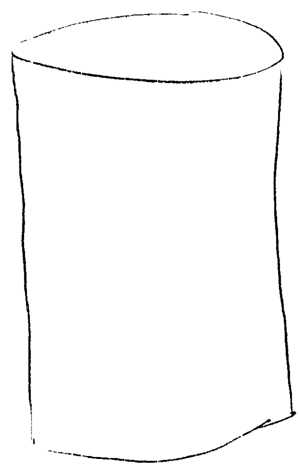
$$= \frac{-\hat{z}\, q^2 a^2}{8\pi^2\epsilon_0} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\infty} \frac{s}{(a^2+s^2)^3}}_{\frac{1}{4a^4}}$$

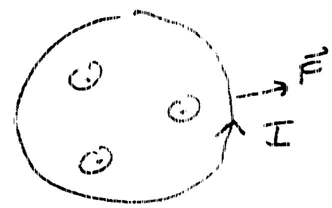
$$\vec{F} = \frac{-\hat{z}\, q^2}{16\pi\epsilon_0 a^2} = \frac{q^2}{4\pi\epsilon_0 (2a)^2}(-\hat{z})$$

Ex Force on Solenoid

$$\vec{B} = B_0 \hat{z}$$

$$B_0 = \mu_0 n I$$
$$= \mu_0 K$$





$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)$$

$$T_{xx} = \frac{1}{\mu_0}\left(\underset{\overset{\shortparallel}{0}}{B_x B_x} - \frac{1}{2}\delta_{ij} B^2 \right) = -\frac{B_0^2}{2\mu_0} = T_{yy}$$

$$T_{zz} = \frac{1}{\mu_0}\left(B_z^2 - \frac{1}{2} B^2 \right) = \frac{B_0^2}{2\mu_0}$$

$$T_{xy} = 0 = T_{xz} \cdots$$

$$\overleftrightarrow{T} = \begin{pmatrix} -\frac{B_0^2}{2\mu_0} & 0 & 0 \\ 0 & -\frac{B_0^2}{2\mu_0} & 0 \\ 0 & 0 & \frac{B_0^2}{2\mu_0} \end{pmatrix}$$

$$\vec{F} = \int_S \overleftrightarrow{T} \cdot d\vec{a}$$

The outward normal ~~for~~ the universe outside the solenoid is $-\hat{s}$, $d\vec{a} = -\hat{s}$. The pressure, force per unit area when $\hat{s} = \hat{x}$ is $\underbrace{\overleftrightarrow{T} \cdot (-\hat{x})}_{''} da$, or

$$f_x = -T_{xx} = \frac{B_0^2}{2\mu_0}$$

We know how to work out the magnetic pressure,

$$d\vec{F} = I\, d\vec{l} \times \vec{B}$$

Force per unit area is $KB = nIB$

That was a little too quick –

The force on a ring of current of length L is

$$F = IL\,B = 2\pi R\, I B$$

If the current I is a current density K flowing through a thickness dz, the $I = K\,dz$ and the force becomes

$$F = 2\pi R\, K\, dz\, B$$

The force per unit area is then

$$\frac{F}{2\pi R\, dz} = K B$$

But remember we have to use the average field of the solenoid $B_{ave} = \frac{1}{2}(B + 0) = B/2$

$$B = \mu_0 K$$

$$\frac{F}{Area} = K\, B/2$$

$$= \frac{B^2}{2\mu_0}$$

What is amazing is that we initially found this ~~By~~ without any information about the currents