Coulomb's Law

We would now like to find the electric field of a point particle with charge q. The charge density is

$$\rho = q\,\delta^3(\vec{r}\,'')$$

where $\vec{r}\,' =$ location of q.

Maxwell's Eqns

$$\nabla \cdot \vec{E} = \frac{q\,\delta^3(\vec{r}\,'')}{\varepsilon_0}$$

$$\nabla \times \vec{E} = 0$$

We know $\nabla \cdot \dfrac{\hat{r}''}{r''^2} = 4\pi\,\delta^3(\vec{r}\,'')$

so try $\vec{E} = C\,\dfrac{\hat{r}''}{r''^2}$

$$\nabla \cdot \vec{E} = 4\pi C\,\delta^3(\vec{r}\,'') = \frac{q}{\varepsilon_0}\,\delta^3(\vec{r}\,'')$$

$$C = \frac{q}{4\pi\varepsilon_0}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r''^2}\,\hat{r}''$$

<u>Coulomb's Law</u> The electric field at a point $\vec{r}$ due to a point electric charge q at point $\vec{r}'$ is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r''^2}\hat{r}'' = \frac{Kq}{r''^2}\hat{r}''$$

$$\vec{r}'' = \vec{r}' - \vec{r} \qquad K = 8.99\times10^9\ Nm^2/C^2$$

<u>Note</u> Since $\nabla\left(\frac{1}{r''}\right) = -\frac{\hat{r}''}{r''^2}$

$$V = \frac{q}{4\pi\varepsilon_0 r''} + C$$

Choose reference point at ∞, $V(\infty) = 0 \Rightarrow C = 0$

<u>Electric Potential of Point Charge</u> - The electric potential at the point $\vec{r}$ due to a point charge at the point $\vec{r}'$ is

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r''} = \frac{Kq}{r''}$$

By linear superposition, we can build up the fields of complicated charge distributions by summing simple distributions

$$\vec{E}_p = \sum_i \frac{q_i}{4\pi\varepsilon_0 r_{iP}^2}\hat{r}_{iP}$$

$$= \int_V \frac{\rho(\vec{r}\,')\,d\tau'}{4\pi\varepsilon_0 r''^2}\hat{r}'' = \int_V \frac{\rho(\vec{r}\,')\,d\tau'}{4\pi\varepsilon_0 r''^2}\hat{r}''$$

$$= \int_S \frac{\sigma(\vec{r}\,')\,da'}{4\pi\varepsilon_0 r''^2}\hat{r}''$$

$$= \int_c \frac{\lambda(\vec{r}\,')\,d\ell'}{4\pi\varepsilon_0 r''^2}\hat{r}''$$

Integrate or sum over sources, the ' coordinate

From the additivity of potential inherited from linear superposition.

$$V(\vec{r}) = \sum_i \frac{q_i}{4\pi\epsilon_0 r''}$$

$$= \int_V \frac{\rho(\vec{r}')\, d\tau'}{4\pi\epsilon_0 r''}$$

$$= \int_S \frac{\sigma(\vec{r}')\, da'}{4\pi\epsilon_0 r''}$$

$$= \int_C \frac{\lambda(\vec{r}')\, d\ell'}{4\pi\epsilon_0 r''}$$

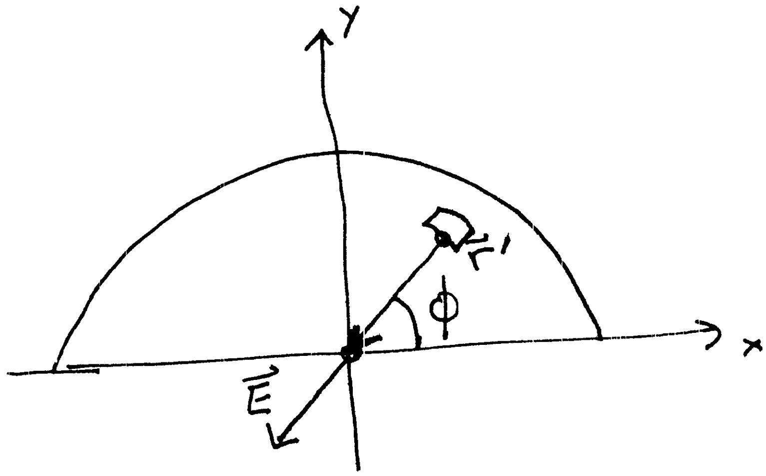
Mechanics

Once the field is known, the force on a charge Q placed at $\vec{r}$ is immediate

$$\vec{F}(\vec{r}) = Q\,\vec{E}(\vec{r})$$

Once the potential is known, the change in potential energy is immediate $\Delta U_{AB} = Q\left(V(\vec{r}_B) - V(\vec{r}_A)\right)$

Ex Calculate $\vec{E}$ for the point at the origin of a uniformly charged half-circle of Radius R s.t. $\sigma = 0$, $y < 0$.



$$\vec{r} = (0,0,0) \qquad \vec{r}\,' = s'\hat{s}'$$

$$\vec{r}\,'' = \vec{r} - \vec{r}\,' = -s'\hat{s}'$$

$$\vec{E}_0 = \int_S \frac{\sigma(\vec{r}\,')\,da'}{4\pi\varepsilon_0 r''^2}\hat{r}'' = \int_S \frac{\sigma(\vec{r}\,')\,da'}{4\pi\varepsilon_0 r''^2}\hat{r}''$$

$$da' = ds'\,s'\,d\phi'$$

$$r'' = s' \qquad \hat{r}'' = -\hat{s}'$$

$$\vec{E}_0 = \frac{-\sigma}{4\pi\varepsilon_0}\int_S \frac{ds'\,s'\,d\phi'}{s'^2}\hat{s}'$$

$$\vec{E}_0 = \frac{-\sigma}{4\pi\varepsilon_0} \int_S \frac{ds'\,d\phi'}{s'} \hat{s}'$$

$$= \frac{-\sigma}{4\pi\varepsilon_0} \int_0^{\pi} d\phi' \int_0^{R} \frac{ds'}{s'} \hat{s}'$$

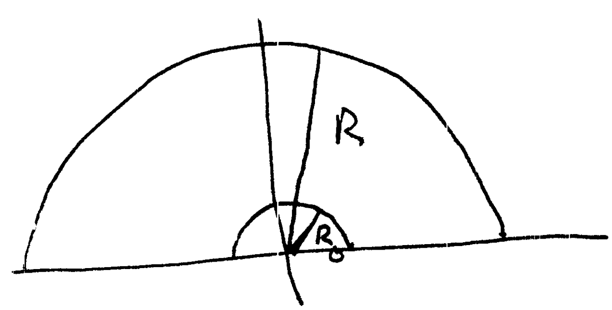
<u>Problem</u> $\hat{s}'$ changes direction as we integrate; express it in terms of $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ (Note, $\hat{x}' = \hat{x}$)

$$\hat{s}' = \cos\phi' \hat{x}' + \sin\phi' \hat{y}'$$

$$\vec{E}_0 = \frac{-\sigma}{4\pi\varepsilon_0} \hat{x} \int_0^{\pi} d\phi' \int_{R_0}^{R} \frac{ds'}{s'} \cos\phi'$$

$$- \frac{\sigma}{4\pi\varepsilon_0} \hat{y} \int_0^{\pi} d\phi' \int_{R_0}^{R} \frac{ds'}{s'} \sin\phi'$$

I introduced a lower limit to R because the integral was singular at origin



$$\int_0^{\pi} \cos\phi' d\phi' = 0$$

$$\int_0^{\pi} \sin\phi\, d\phi' = -\cos\phi'\Big|_0^{\pi} = -(-1) - (-1) = 2$$

$$\vec{E}_0 = \frac{-2\sigma}{4\pi\varepsilon_0} \hat{y} \int_{R_0}^{R} ds' \frac{1}{s'}$$

$$= \frac{-\sigma}{2\pi\varepsilon_0} \ln\left(\frac{R}{R_0}\right) \hat{y}$$

Ex Compute the potential of the same system at origin.

Sln

$$V(0) = \int_S \frac{\sigma(\vec{r}')\, da'}{4\pi\varepsilon_0 r'} \qquad da' = s' ds' d\phi' \quad r' = s'$$

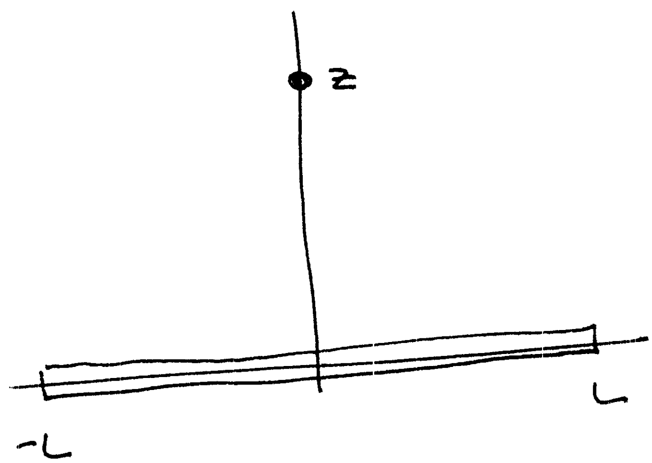
$$= \frac{\sigma}{4\pi\varepsilon_0} \underbrace{\int_0^{\pi} d\phi'}_{\pi} \underbrace{\int_{R_0}^{R} \frac{s' ds'}{s'}}_{R - R_0}$$

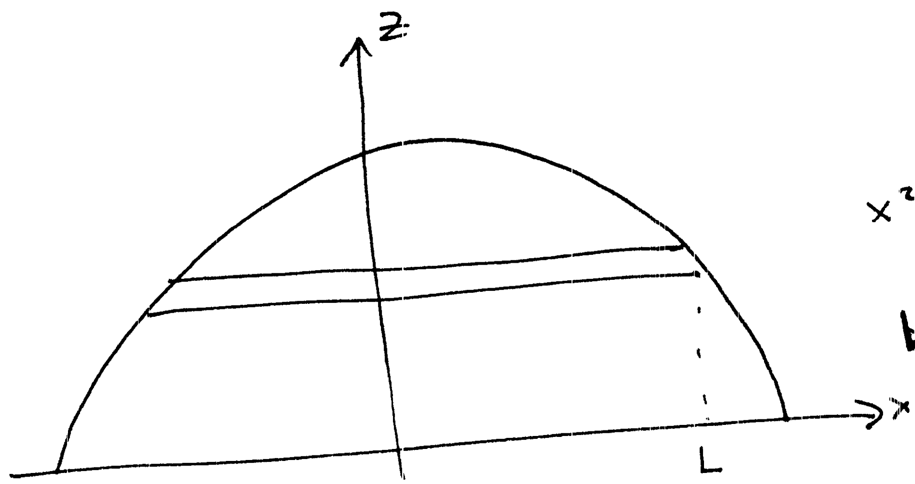
$$= \frac{\sigma}{4\varepsilon_0}(R - R_0)$$

That was weird, better check it.

The field of a finite wire is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2+L^2}} \hat{z}$$





$x^2 + z^2 = R^2$

$L = \sqrt{R^2 - z^2}$

$\lambda = \sigma dz$

$$\vec{E} = \int_0^R dE = \frac{2}{4\pi\epsilon_0} \int_0^R \frac{\sigma dz \sqrt{R^2 - z^2}}{z\sqrt{z^2 + (R^2 - z^2)}} (-\hat{z})$$

$$= \frac{2\sigma(-\hat{z})}{4\pi\epsilon_0 R} \int_0^R \frac{dz\sqrt{R^2-z^2}}{z}$$

$> int\left(\frac{sqrt\left(R^2 - z^2\right)}{z}, z\right);$

$$\sqrt{R^2 - z^2} - \frac{R^2 \ln\left(\frac{2 R^2 + 2\sqrt{R^2}\sqrt{R^2 - z^2}}{z}\right)}{\sqrt{R^2}} \qquad (1)$$

$> int\left(\frac{sqrt\left(R^2 - z^2\right)}{z}, z = 0 ..R\right);$

$$\mathrm{signum}(R)\,\mathrm{csgn}(\mathrm{signum}(R))\,\infty \qquad (2)$$

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