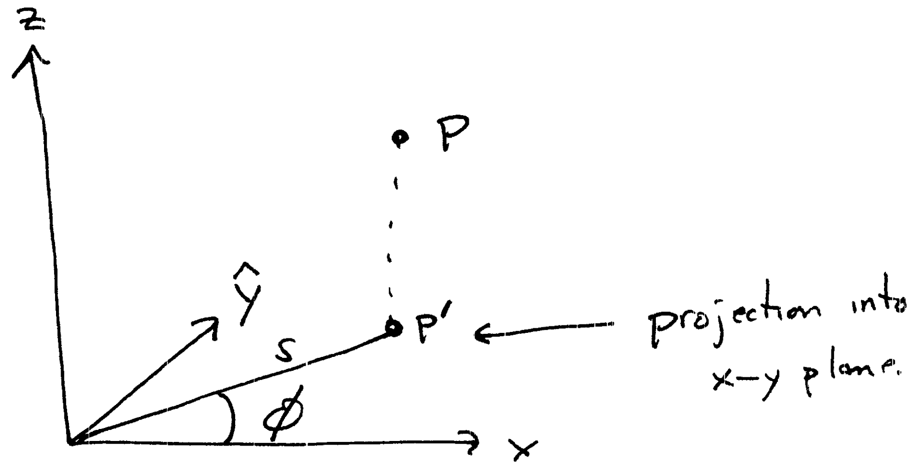
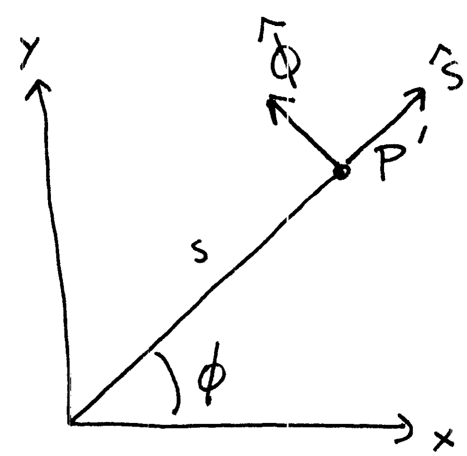
Curvilinear Coordinates

Cartesian $\hat{x} \times \hat{y} = \hat{z}$ (right handed triple)

Cylindrical $\hat{s} \times \hat{\phi} = \hat{z}$ (right handed triple)



In x–y plane



Volume element $d\tau = (ds)(s\,d\phi)\,dz$

Path element $d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

$$\vec{r}_p = x\hat{x} + y\hat{y} + z\hat{z}$$

$$= f_1\hat{s} + f_2\hat{\phi} + f_3\hat{z}$$

$$\cancel{x = \vec{r}_p} \cdot \qquad f_1 = \hat{s}\cdot\vec{r}_p$$

$$= x\,\hat{s}\cdot\hat{x} + y\,\hat{s}\cdot\hat{y} + z\,\hat{s}\cdot\hat{z}$$

Transformation Equations

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{z} = \hat{z}$$

$$\hat{s}\cdot\hat{x} = \cos\phi \qquad \hat{s}\cdot\hat{y} = \sin\phi \qquad \hat{s}\cdot\hat{z} = 0$$

$$f_1 = x\cos\phi + y\sin\phi$$

$$|f_1| = \sqrt{x^2+y^2} \equiv s$$

why $\cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$ $\sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$

$$\Rightarrow \quad f_1 = \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

- Div, grad, curl in cylindrical coordinates

$$\nabla \neq \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial z}\right)$$ because

$\hat{s}, \hat{\phi}$ change with position. $\Rightarrow$ The coordinate unit vectors contribute to the gradient.

- Simply look up correct form. (front cover).

<u>Ex</u> Compute $\nabla \cdot (s\hat{\phi})$

$$\vec{A} = (A_s, A_\phi, A_z) = A_s\hat{s} + A_\phi\hat{\phi} + A_z\hat{z}$$

$$A_s = 0, \quad A_\phi = s, \quad A_z = 0$$

<u>From front cover</u>

$$\nabla \cdot \vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\nabla \cdot (s\hat{\phi}) = \frac{1}{s}\frac{\partial}{\partial s}(s \cdot 0) + \frac{1}{s}\frac{\partial s}{\partial \phi} + \frac{\partial 0}{\partial z}$$

$$= 0$$

<u>Don't differentiate twice</u> - The derivative formulas take care of the change in $\hat{\phi}$, $\hat{s}$, $\hat{r}$, $\hat{\theta}$ so don't differentiate them again.

$\Rightarrow$ When in doubt, use cartesian.

<u>Ex</u> Is $\nabla \cdot \hat{s} = 0$ (it would be if $\hat{s} \to \hat{x}$)

$$\hat{s} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0 \right)$$

since $\vec{s} = (x, y, 0)$

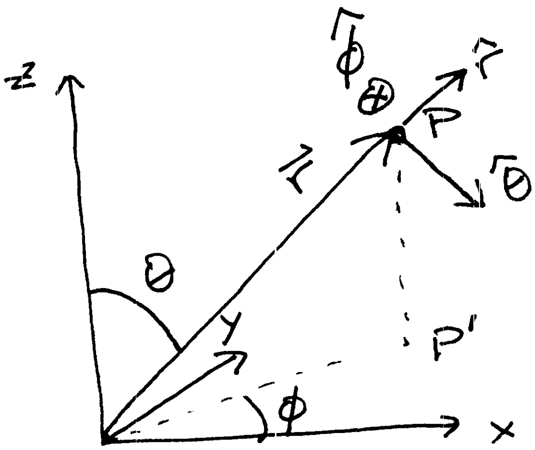
$$\nabla \cdot \hat{s} = \left(\frac{\partial}{\partial x} \frac{x}{\sqrt{x^2+y^2}} + \frac{\partial}{\partial y} \frac{y}{\sqrt{x^2+y^2}} + 0 \right)$$

$$\neq 0$$

Spherical Coordinates

$$\vec{r}_p = (x, y, z) = r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

$$r = |\vec{r}_p| = \sqrt{x^2 + y^2 + z^2}$$



$\hat{r} \times \hat{\theta} = \hat{\phi}$ right handed triple

Line element $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

Volume Element $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$= (dr)(r\,d\theta)(r\sin\theta\,d\phi)$

Ex Let's work out A_θ at the top of the page, dot both sides with $\hat{\theta}$

$$A_\theta = x\,\hat{x}\cdot\hat{\theta} + y\,\hat{y}\cdot\hat{\theta} + z\,\hat{z}\cdot\hat{\theta}$$

Look up transformations - Back cover

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{x}\cdot\hat{\theta} = \cos\theta\cos\phi$$

$$\hat{y}\cdot\hat{\theta} = \cos\theta\sin\phi$$

$$\hat{z}\cdot\hat{\theta} = -\sin\theta$$

$$A_\theta = x\cos\theta\cos\phi + y\cos\theta\sin\phi - z\sin\theta$$

Ex Find $\cancel{\nabla^2(r^2\theta)}$ $\nabla^2 r$

Front cover

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$\nabla^2 r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial r}{\partial r}\right) = \frac{2r}{r^2} = \frac{2}{r}$$