Cylindrical Coordinates

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Trivial Solutions $\quad 1, \ln(s), \phi, z$

Axially Radial Does not depend on ϕ, z

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

$$V = 1, \ \ln(s)$$

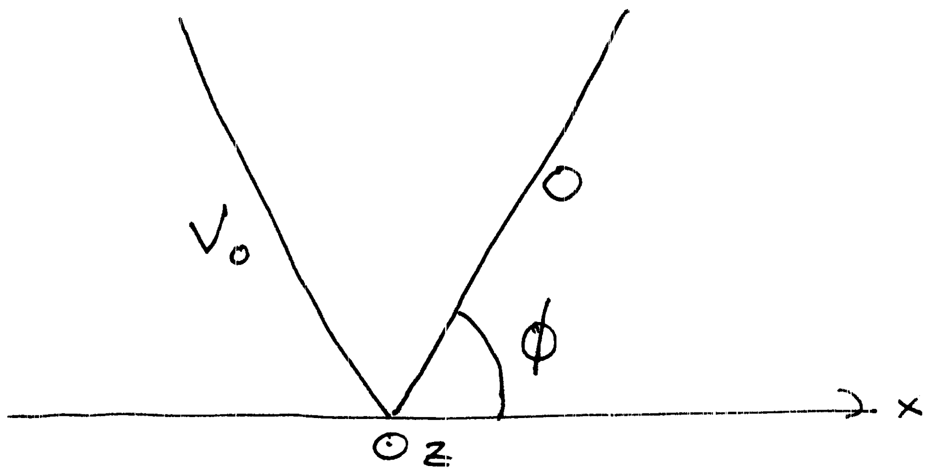
General Solution

$$V = a_1 + a_2 \ln(s)$$

Azimuthal Does not depend on s, z.

$$V = V_0 \phi + C$$

Ex Compute field in wedge where one face is held at $V(60°) = 0$ and the other face is held at $V(120°) = V_0$.



Sln V does not depend on s, z.

$$V = a_1 + a_2 \phi$$

$$V(60) = a_1 + a_2 \cdot \frac{\pi}{3} = 0$$

$$V(120) = a_1 + a_2 \frac{2\pi}{3} = V_0$$

$$\frac{\pi}{3} a_2 = V_0$$

$$a_2 = \frac{3V_0}{\pi}$$

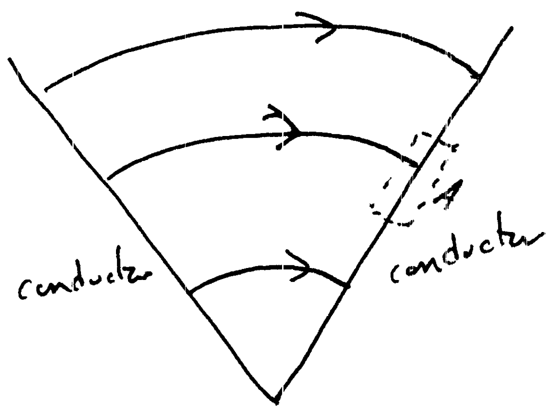
$$a_1 = - a_2 \frac{\pi}{3} = - \left(\frac{3V_0}{\pi}\right) \frac{\pi}{3} = -V_0$$

$$V = -V_0 + \frac{3V_0}{\pi}\phi$$ Solution

Electric Field

$$\vec{E} = -\nabla V = -\frac{3V_0}{\pi s}\hat{\phi}$$

Charge Density



$$\frac{A\sigma}{\varepsilon_0} = \Phi = \vec{E}\cdot\hat{n}A$$

$$\sigma = \varepsilon_0 \vec{E}\cdot\hat{\phi} = -\frac{3V_0\varepsilon_0}{\pi s}$$

Infinitely Long Cylinders (no z)

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Separation $\quad V = S(s)\,\Phi(\phi)$

$$\underbrace{\frac{1}{S(s)}\, s\,\frac{\partial}{\partial s}\left(s\frac{\partial \cancel{S}}{\partial s}\right)}_{= k^2} + \underbrace{\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi}{\partial \phi^2}}_{= -k^2} = 0$$

Solve for Φ

$$\frac{d^2\Phi}{d\phi^2} + k^2\Phi = 0$$

$$\Phi = \sin k\phi,\ \cos k\phi$$

To be continuous, at $\phi = 0$, $\phi = 2\pi$, k must be an integer, $k = n$.

$$\Phi = \sin n\phi,\ \cos n\phi$$

Solve s

$$s\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) - n^2 S = 0$$

Try $S(s) = s^a$

$$s\frac{\partial}{\partial s}\left(a s^a\right) - n^2 S = 0$$

$$-a^2 s^a - n^2 s^a = 0$$

$$a = \pm n$$

Solutions

$$1 \qquad \ln \rho \qquad \phi$$

$$\rho^n \cos n\phi \qquad \rho^{-n} \cos n\phi$$

$$\rho^n \sin n\phi \qquad \rho^{-n} \sin n\phi$$

Orthoganolity

$$\int_0^{2\pi} \sin n\phi \, \sin m\phi \, d\phi = \begin{cases} 0 & m \neq n \\ \pi & n = m \end{cases}$$

Ex An infinite cylinder has a potential $V(\phi) = V_0 \cos^2\phi$ established on the surface. Compute field outside cylinder

Sln

$$V = \sum A_n s^n \cos n\phi + B_n s^n \sin n\phi$$
$$+ C_n s^{-n} \cos n\phi + D_n s^{-n} \sin n\phi$$

Since outside, V must be finite as $s \to \infty$

$\Longrightarrow A_n, B_n = 0$

Work on given potential, trig identity

$$\cos^2\phi = \frac{1}{2} + \frac{1}{2}\cos 2\phi$$

Apply, B.C.

$$V(a, \phi) = \frac{V_0}{2} + \frac{V_0}{2}\cos 2\phi$$

$$= \sum_n C_n a^{-n} \cos n\phi + D a^{-n} \sin n\phi$$

The $\frac{V_0}{2}$ term can be satisfied by one of the trivial solutions. So we need the expansion to satisfy $\frac{V_0}{2}\cos 2\phi$. We can use Fourier's Trick, or simply the fact that the $\cos n\phi$ are orthogonal

$$\frac{V_0}{2}\cos 2\phi = \sum_n C_n a^{-n}\cos n\phi + D_n a^{-n}\sin n\phi$$

Orthogonality implies $D_i = 0$, $C_i = 0$ except C_2

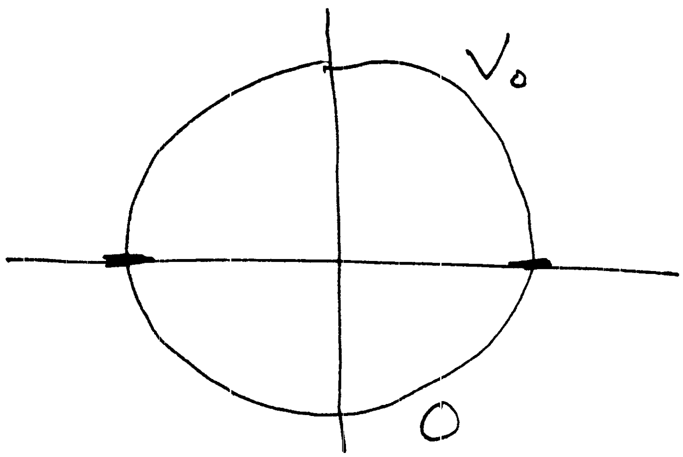
$$C_2 a^{-2} = \frac{V_0}{2}$$

$$C_2 = \frac{V_0 a^2}{2}$$

Therefore the potential is

$$V(s,\phi) = \frac{V_0}{2} + \frac{V_0 a^2}{2s^2}\cos 2\phi$$

Ex A little harder, infinite cylinder with top half at V_0 and bottom half at zero. Compute field inside.



S/n Since inside, s^{-n} terms blow up.

$$V(s,\phi) = \sum A_n s^n \cos n\phi + B_n s^n \sin n\phi$$

Orthogonality

$$\int_0^{2\pi} \cos n\phi \cos m\phi \, d\phi = \delta_{nm} \pi \quad \text{if } n,m>0$$

$$= 2\pi \quad \text{if } n=m=0$$

$$\int_0^{2\pi} \sin n\phi \sin m\phi \, d\phi = \delta_{nm} \pi$$

Boundary Conditions

$$V(a,\phi) = \begin{cases} 0 < \phi < \pi & V = V_0 \\ \pi < \phi < 2\pi & V = 0 \end{cases}$$

$$= \sum_n \left(A_n a^n \cos n\phi + B_n a^n \sin n\phi \right)$$

$$\left.\begin{aligned} \int_0^{2\pi} V(a,\phi) \cos m\phi \, d\phi &= \sum_n A_n a^n \pi \delta_{nm} \\ &= A_m a^m \pi \end{aligned}\right] \text{, f } n,m \neq 0$$

$$\int_0^{\pi} V_0 \cos m\phi \, d\phi$$

$$A_m a^m \pi = \frac{V_0}{m} \sin m\phi \Big|_0^{\pi} = 0$$

Likewise,

$$B_m a^m \pi = \int_0^{\pi} V_0 \sin m\phi \, d\phi = -\frac{V_0}{m} \cos m\phi \Big|_0^{\pi}$$

$$= \begin{cases} +\dfrac{2V_0}{m} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

The final case is if $n=0$, $m=0$, then

$\cos n\phi = 1$ and

$$\int_0^{2\pi} \cos m\phi \, V(a,\phi)\, d\phi = \sum_n A_n a^n \int_0^{2\pi} \cos m\phi \cos n\phi \, d\phi$$

$$+ \; 0$$

If $n=m=0$, $\int_0^{2\pi} \cos n\phi \cos m\phi \, d\phi = \int_0^{2\pi} d\phi = 2\pi \delta_{nm}$

so

$$2\pi A_0 a^0 = \int_0^{2\pi} \cos 0\phi \, V(a,\phi)\, d\phi = \int_0^{2\pi} V(a,\phi)\, d\phi$$

$$= \int_0^{\pi} V_0 \, d\phi = \pi V_0$$

$$A_0 = \frac{V_0}{2}$$

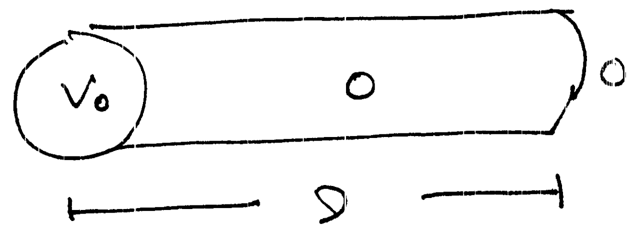
$$B_m = \begin{cases} 0 & m \text{ even} \\ \dfrac{+2V_0}{m\pi a^m} & m \text{ odd} \end{cases}$$

Full Solution

$$V(s,\phi) = \sum_{n\,\text{odd}} \frac{+2V_0}{n\pi a^n} s^n \sin n\phi \quad + \frac{V_0}{2}$$

Full Solution – Cylindrical – depends on s, ϕ, z.

Ex

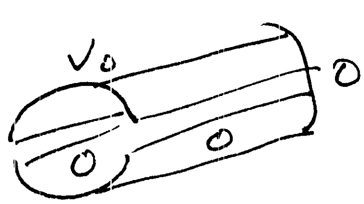


General Solution

$$\left[J_\nu(sk) + N_\nu(sk) \right] \times \left[\sin \nu\phi + \cos \nu\phi \right] \times \left[e^{kz} + e^{-kz} \right]$$

Bessel + Neumann Functions.

If we did,

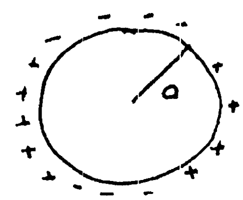


we get modified Bessel functions.

$\boxed{\text{Note old notation, let} \quad \rho \to s \quad \hat{\rho} \to \hat{s}}$

__Ex__ An infinitely long cylinder has a potential at the surface of $V(a,\phi) = V_0 \cos 2\phi$

Compute field everywhere.



__Solutions__ $1, \ \ln \rho, \ \phi$

$$\rho^n \cos n\phi \qquad \rho^{-n} \cos n\phi$$

$$\rho^n \sin n\phi \qquad \rho^{-n} \sin n\phi$$

__Inside__

$$V_i(\rho, \phi) = \sum_n A_n \rho^n \cos n\phi + B_n \rho^n \sin n\phi$$

$$V(a,\phi) = V_0 \cos 2\phi = \sum A_n a^n \cos n\phi + B_n a^n \sin n\phi$$

$$\Longrightarrow \quad A_n, B_n = 0 \quad \text{except} \quad A_2 = \frac{V_0}{a^2}$$

$$V_i(\rho, \phi)_{inside} = \frac{V_0}{a^2} \rho^2 \cos 2\phi$$

<u>Outside</u>

$$V_o(\rho, \phi) = \sum A_n \rho^{-n} \cos n\phi + B_n \rho^{-n} \sin n\phi$$

$$V(a, \phi) = V_0 \cos 2\phi$$

$A_n, B_n = 0$ except

$$A_2 = a^2 V_0$$

$$V_o(\rho, \phi) = \frac{V_0 a^2}{\rho^2} \cos 2\phi$$

Field Inside

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi}$$

$$\vec{E}_{inside} = -\nabla V$$

$$= \frac{-2V_0}{a^2}\rho\cos 2\phi\,\hat{\rho} + \frac{2V_0}{a^2}\rho\sin 2\phi\,\hat{\phi}$$

$$\vec{E}_{outside} = \frac{2V_0 a^2}{\rho^3}\cos 2\phi\,\hat{\rho} + \frac{2V_0 a^2}{\rho^3}\sin 2\phi\,\hat{\phi}$$

Surface Charge   Use Pill box at surface

$$\sigma = \left(\vec{E}_{outside} - E_{inside}\right)\cdot\hat{\rho}\,\varepsilon_0$$

$$\sigma = \frac{2V_0}{a}\cos 2\phi - \left(-\frac{2V_0}{a}\cos 2\phi\right)$$

$$= \frac{4V_0}{a}\cos 2\phi$$