Work, Force, Energy, and Capacitance

Gauss' Law with Electric Displacement

$$\nabla \cdot \vec{D} = \rho_f \qquad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

If the dielectric is linear

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}$$

$$\varepsilon_r = (1 + \chi_e)$$

Electrostatic Boundary Conditions

$$\nabla \cdot \vec{D} = \rho_f \implies \vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = \sigma_f$$

$$\nabla \times \vec{E} = 0 \implies \vec{E}_2 \cdot \hat{t} = \vec{E}_1 \cdot \hat{t}$$

If the dielectric completely fills the fieldspace, and there are no interfaces then

$$\nabla \cdot \vec{D} = \rho_f \ , \quad \nabla \times \vec{D} = 0$$

which are the same equations we would have without dielectric. Solve equations ignoring dielectric to yield $\vec{D} = \varepsilon_0 \vec{E}_0$ where $\vec{E}_0$ is the field that would exist if the free charge were held in place but the dielectric vanished

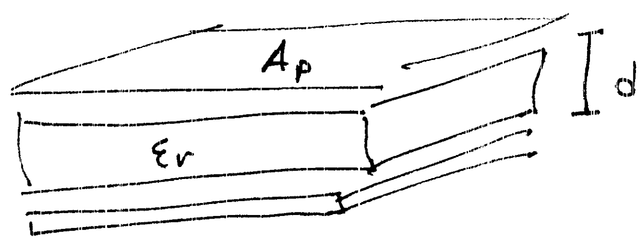
This must also be the $\vec{D}$ with the dielectric present, $\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E}_0$

$$\vec{E} = \frac{\vec{E}_0}{\varepsilon_r}$$

where $\vec{E}$ is the field with the dielectric.

The next effect of the dielectric is to reduced the field by ε_r. This naturally affects the capacitance.

Ex Parallel Plate Capacitor Filled with dielectric



S/n If no fringing, dielectric completely fills field space

The electric field ignoring the dielectric is $\sigma/\varepsilon_0 = \dfrac{Q}{\varepsilon_0 A_p}$ where I have placed $+Q$ on top plate, $-Q$ on bottom plate.

The potential difference between the plates without the dielectric is

$$\Delta V = -\int_0^d \vec{E} \cdot d\vec{l} = |\vec{E}_0| d = \frac{\sigma d}{\varepsilon_0}$$

The capacitance is by definition

$$C_0 = \frac{Q}{|\Delta V|} = \frac{\sigma A}{|\Delta V|} = \frac{\varepsilon_0 A}{d}$$

With the dielectric, the field is reduced by ε_r.

$$E_K = \frac{E_0}{\varepsilon_r} = \frac{\sigma}{\varepsilon_r \varepsilon_0}$$

The potential difference is still

$$\Delta V = -\int_0^d \vec{E} \cdot d\vec{l} = |E_K| d = \frac{\sigma d}{\varepsilon_r \varepsilon_0}$$

$$C_K = \frac{Q}{|\Delta V|} = \frac{\sigma A}{|\Delta V|} = \varepsilon_r \left(\frac{\varepsilon_0 A}{d} \right) = \varepsilon_r C_0$$

If the dielectric completely fills the fieldspace, the capacitance is increased by ε_r. This is a general result for a capacitor filled with a single dielectric.

<u>Energy</u> The work to charge the capacitor is still

$$W = \int dW = \int_0^Q \Delta\cancel{V}\, dQ = \int_0^Q \frac{Q\,dQ}{C}$$

$$= \frac{1}{2}\frac{Q^2}{C} = U = \frac{1}{2}C(\Delta V)^2$$

since energy depends on ΔV which enters the definition of capacitance.

Returing to our parallel-plate capacitor. Without the dielectric, the energy density was

$$u = \frac{U}{\text{Volume}} = \frac{\frac{1}{2}C(\Delta V)^2}{A\,d}$$

$$= \frac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)(\Delta V)^2 \cdot \frac{1}{Ad}$$

$$= \frac{1}{2}\varepsilon_0\left(\frac{\Delta V}{d}\right)^2 = \frac{1}{2}\varepsilon_0 E^2$$

With the dielectric, the energy density becomes

$$u = \frac{U}{\text{volume}} = \frac{\frac{1}{2} C (\Delta v)^2}{A d}$$

$$= \frac{1}{2} \left(\varepsilon_r \frac{\varepsilon_0 A}{d} \right) \left(\Delta v^2 \right) \frac{1}{A d}$$

$$= \frac{1}{2} \varepsilon_r \varepsilon_0 E^2 = \frac{1}{2} \varepsilon E^2$$

~~If the dielectric was not linear, this~~

This can be re-written as

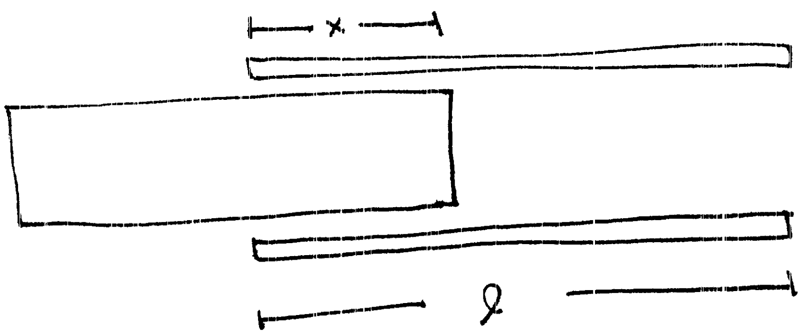
$$u = \frac{1}{2} \vec{E} \cdot \vec{D}$$

the energy density of the electric field in the presense of a dielectric.

Force on Dielectric

As dielectric is inserted into a capacitor, the energy of the system changes, implying a force.

$$F = -\frac{dW}{dx} = -\frac{dU}{dx}$$



Let other dimension be l.

Two Cases

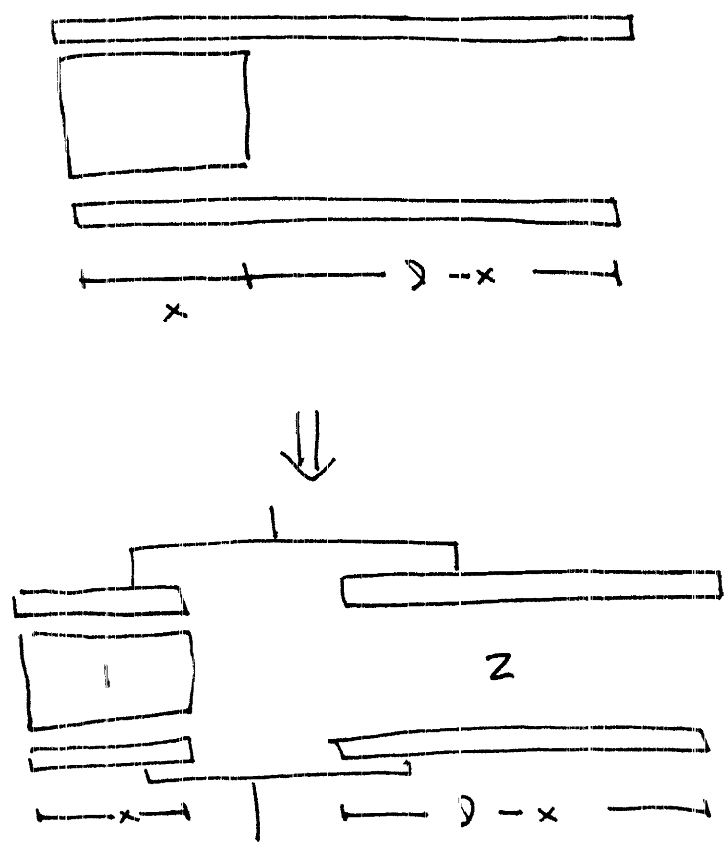
I. Connected to battery (ΔV = constant)

II. Charged, but disconnected (Q = constant)

Case II $\quad Q$ = constant

$$F = -\frac{dU}{dx} = -\frac{d}{dx}\frac{Q^2}{2C} = -\frac{Q^2}{2}\frac{d}{dx}\frac{1}{C}$$

$$= \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$



~~Paralell~~ Parallel Capacitors

$$C_{eq} = C_1 + C_2 = \frac{A_1 \varepsilon_0 \varepsilon_r}{d} + \frac{A_2 \varepsilon_0}{d}$$

$$= \frac{\ell(x)\, \varepsilon_0 \varepsilon_r}{d} + \frac{\ell(\ell - x)\, \varepsilon_0}{d}$$

$$\frac{dC_{eq}}{dx} = \frac{\ell \varepsilon_0 \varepsilon_r}{d} - \frac{\ell \varepsilon_0}{d} = \frac{\ell \varepsilon_0}{d}(\varepsilon_r - 1)$$

$$= \frac{C_0}{\ell}(\varepsilon_r - 1)$$

where $C_0 = \dfrac{\ell^2 \varepsilon_0}{d}$ the air-filled capacitance

$$C_{eq} = \frac{\ell \varepsilon_0}{d}\left(\varepsilon_r x + \ell - x\right) = C_0\left((\varepsilon_r - 1)\frac{x}{\ell} + 1\right)$$

$$F = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} =$$

$$= \frac{1}{2} Q^2 \frac{C_0/\ell\,(\varepsilon_r - 1)}{C_0^2\left((\varepsilon_r - 1)\frac{x}{\ell} + 1\right)^2} = \frac{1}{2}\frac{Q^2}{C_0 \ell}\frac{\varepsilon_r - 1}{\left((\varepsilon_r - 1)\frac{x}{\ell} + 1\right)^2}$$

$\Rightarrow F > 0$ Dielectric sucked into capacitor

<u>Case II</u> $\Delta V =$ constant - connected to battery

Must consider dielectric / battery system

$$dU_{sys} = dU_{cap} - V dQ_{batt}$$

$$\frac{dU_{cap}}{dx} = \frac{d}{dx}\frac{1}{2} C (\Delta V)^2 = \frac{1}{2}(\Delta V)^2 \frac{dC}{dx}$$

$$dW_{batt} = -V\,dQ$$

$$= -V(V\,dC)$$

$$\frac{dW_{batt}}{dx} = -V^2\frac{dC}{dx}$$

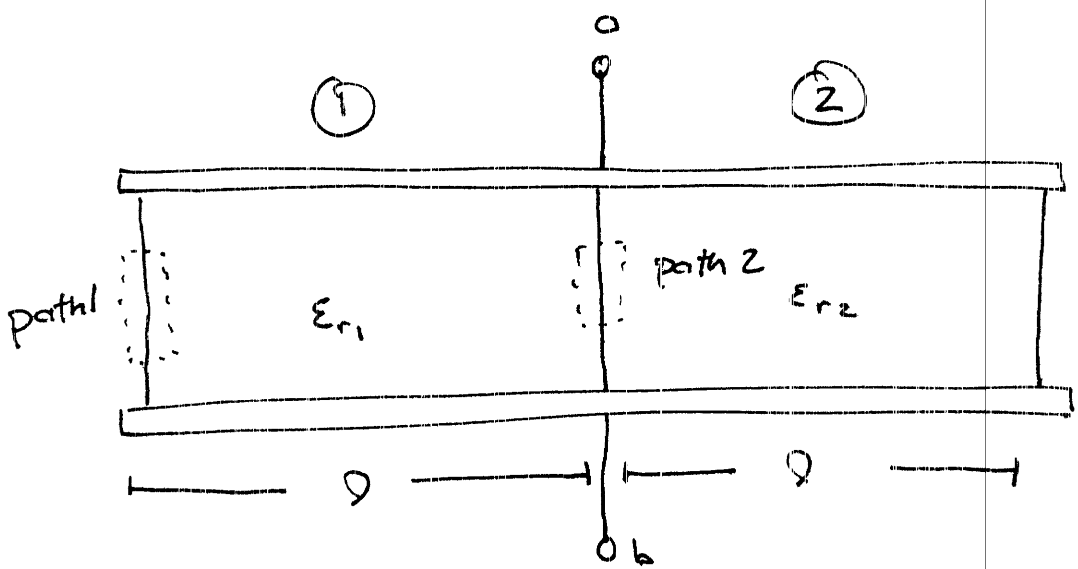
$$\frac{dU_{sys}}{dx} = \frac{1}{2}V^2\frac{dC}{dx} - V^2\frac{dC}{dx}$$

$$= -\frac{1}{2}V^2\frac{dC}{dx}$$

$$F = -\frac{dU_{sys}}{dx} = \frac{1}{2}V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{C_0}{l}(\varepsilon_r - 1)$$

$\Rightarrow$ Force is still positive, dielectric drawn into capacitor.

Let's consider the two dielectric capacitor some more



Let width be w

$$C_{ab} = C_1 + C_2 = \frac{\varepsilon_{r_1}\varepsilon_0 l w}{d} + \frac{\varepsilon_{r_2}\varepsilon_0 l w}{d}$$

Let's charge the capacitor to potentail difference V_0

<u>Electric Field</u>

$$E_1 = \frac{V}{d} \qquad E_2 = \frac{V}{d}$$

E.S. boundary condition satisfied for path 2 but not path 1. If system extends to ∞ fields are exact, if not we must assume no fringing.

The fields are produced by charge densities

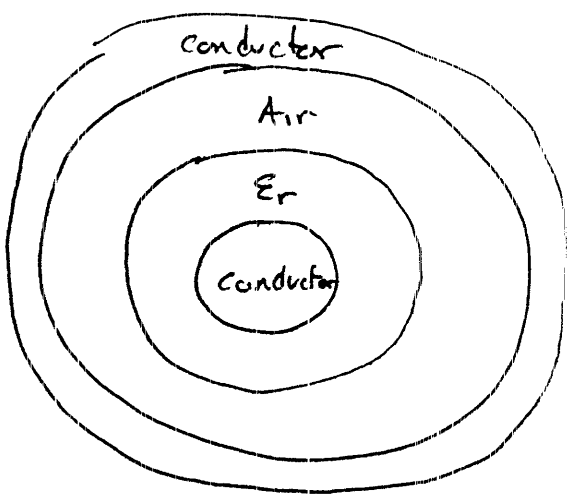
$$E_1 = \frac{\sigma_1}{\varepsilon_0 \varepsilon_{r1}} \qquad E_2 = \frac{\sigma_2}{\varepsilon_0 \varepsilon_{r2}}$$

$$\sigma_1 = \varepsilon_0 \varepsilon_{r1} E_0$$

$$= \varepsilon_0 \varepsilon_{r1} \frac{V}{d} \qquad \sigma_2 = \varepsilon_0 \varepsilon_{r2} \frac{V}{d}$$

Note — $\sigma_1 \neq \sigma_2$

Let's try some other geometries, spherical



Dielectric does not completely fill fieldspace so

$$C \neq \varepsilon_r C_0$$

Introduce Q on inner conductor, the displacement flux is $\Phi_d = 4\pi r^2 D = Q_f = Q$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

<u>Check B. C.</u>

(1) No σ_f, so $\vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = 0$ ✓

(2) Field radial, so $\vec{E}_2 \cdot \hat{t} = \vec{E}_1 \cdot \hat{t}$ ✓

<u>Solution Exact</u>

$$\varepsilon_0 \vec{E}_{air} = \vec{D}, \qquad \varepsilon \vec{E}_{diel} = \vec{D}$$

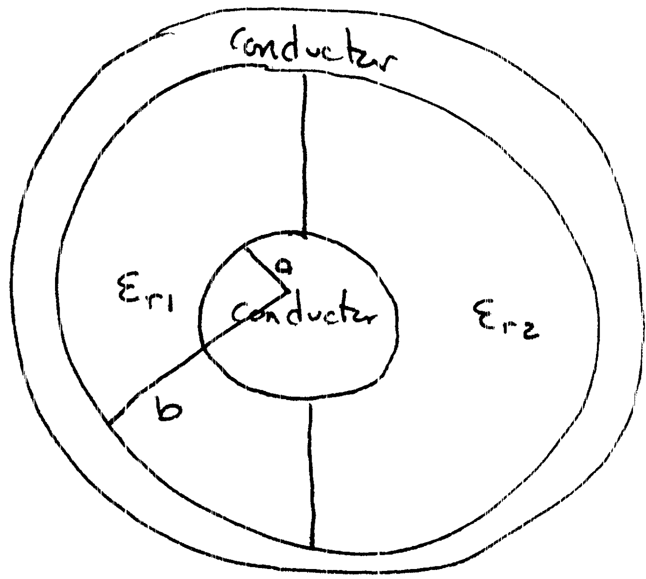
$$\vec{E}_{air} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \vec{E}_{diel} = \frac{Q}{4\pi\varepsilon_0 r^2 \varepsilon_r}\hat{r}$$

$$= \frac{Q}{4\pi\varepsilon r^2}\hat{r}$$

<u>Potential Difference</u>

$$\Delta V_{diel} = -\int \vec{E}_{diel} \cdot d\vec{l} \qquad \Delta V_{air} = -\int \vec{E}_{air} \cdot d\vec{l}$$

What about?



Once again, place $+Q$ on inner conductor, $-Q$ outer.

Note, Q will not be uniformly distributed.

ΔV between the conductors will be the same for all paths between the conductors.

Let V be the potential difference between the shells.

If either dielectric completely filled the airspace, we would find $\vec{D}_1 = \varepsilon_{r1}\varepsilon \vec{E}_1$ or $\vec{D}_2 = \varepsilon_{r2}\varepsilon_0 \vec{E}_2$

$$\vec{E}_1 = \frac{Q_1}{4\pi\varepsilon_0\varepsilon_{r1} r^2}\hat{r} \qquad \vec{E}_2 = \frac{Q_2}{4\pi\varepsilon_0\varepsilon_{r2} r^2}\hat{r}$$

$$\Delta V_1 = -\int \vec{E}_1 \cdot d\vec{l} = \frac{Q_1}{4\pi\varepsilon_0\varepsilon_{r_1}}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Delta V_2 = \frac{Q_2}{4\pi\varepsilon_0\varepsilon_{r_1}}\left(\frac{1}{a} - \frac{1}{b}\right)$$

If $\Delta V_1 = \Delta V_2$, $\quad Q_1 \neq Q_2$

$$V = \Delta V_1 = \frac{4\pi a^2 \sigma_1}{4\pi\varepsilon_0\varepsilon_{r_1}}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\sigma_1}{\varepsilon_0\varepsilon_{r_1}}\left(a - \frac{a^2}{b}\right)$$

$$\sigma_1 = \frac{\varepsilon_0\varepsilon_{r_1} V}{\left(a - a^2/b\right)} \qquad \sigma_2 = \frac{\varepsilon_0\varepsilon_{r_2} V}{\left(a - a^2/b\right)}$$

Note, we can use the fields generated by σ_1 and σ_2 because we satisfy $\nabla\cdot\vec{D} = \rho_f$ and $\nabla\times\vec{E} = 0$ at all boundaries.

$$C = \frac{Q}{V} = \frac{2\pi a^2\sigma_1 + 2\pi a^2\sigma_2}{V}$$