Delta Functions

The dirac delta function is a continuous generalization of the Kronecker delta δ_{ij}

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

The delta function is

$$\delta_{xx'} = \begin{cases} 0 & \text{if } x \neq x' \\ \infty \text{ (1 in a special way)} & \text{if } x = x' \end{cases}$$

We will write $\delta_{xx'}$ as $\delta(x-x')$ and define $\delta(x-a)$ as a "distribution" s.t.

$$\delta(x-a) = 0 \quad \text{if } x \neq a$$

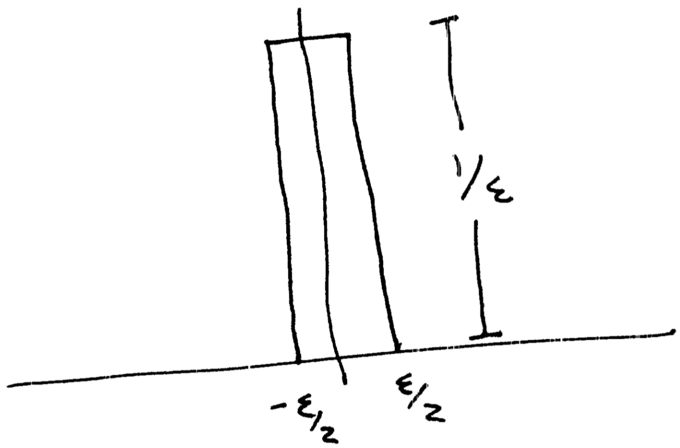
and

$$\int_{r_1}^{r_2} \delta(x-a)\,dx = \begin{cases} 0 & \text{if } a \notin [r_1, r_2] \\ 1 & \text{if } a \in [r_1, r_2] \end{cases}$$

* Note the delta function is not a real function. It only makes sense within an integral.

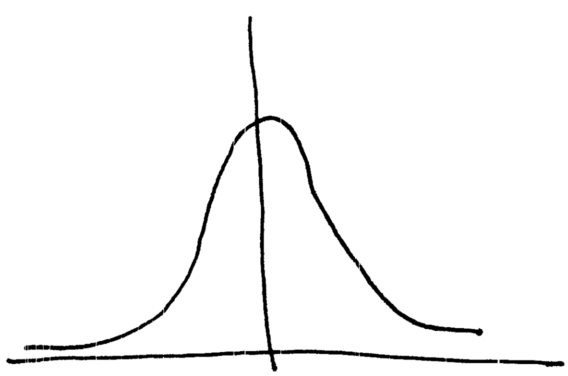
You can imagine the delta function as the limit of a number of functions

$$\delta(x) = \lim_{\varepsilon \to 0} f_\varepsilon = \begin{cases} 0 & x < -\varepsilon/2 \\ 1/\varepsilon & -\varepsilon/2 < x < \varepsilon/2 \\ 0 & x > \varepsilon/2 \end{cases}$$



This always has area 1, but becomes infinitely thin in the limit $\varepsilon \to 0$.

One can also picture the limit as a Gaussian becomes infinitely thin.



$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{2\sqrt{\pi|\varepsilon|}} e^{-x^2/4|\varepsilon|}$$

Delta functions make integration easy

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

<u>Ex</u>

$$\int_{0}^{2\pi} \sin^3(x)\, x^5\, \delta(x-\pi)\,dx$$

$$= \sin^3(\pi)\,\pi^5 = 0$$

To prove something about a delta function you have to show

$$\int F[\delta(x)]\,f(x)\,dx = \int G[\delta(x)]\,f\,dx$$

for all x. $\Rightarrow$ $F[\delta(x)] = G[\delta(x)]$

Ex Show $x \frac{d\sigma}{dx} = -\sigma(x)$

S/n Must show for all f

$$\int_{-\infty}^{\infty} f(x) \left[x \frac{d\sigma(x)}{dx} \right] dx = \int_{-\infty}^{\infty} f(x)\, x \frac{d\sigma}{dx}\, dx$$

$$= f(x)\, x\, \sigma(x) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} \sigma(x) \frac{d}{dx}\left(x f(x)\right) dx$$

First term zero since $\sigma(\infty) = 0$.

Integration by parts.

Second term -

$$- \int_{-\infty}^{\infty} \sigma(x) \left[f(x) + x \frac{df}{dx} \right] dx$$

$$= -f(0) - 0 \left.\frac{df}{dx}\right|_0 = -f(0)$$

$$= - \int \sigma(x) f(x)\, dx$$

for all f ✓

Three dimensional delta functions

$$\delta^3(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$\int_{\text{volume}} \delta^3(\vec{r}-\vec{r}\,')\,d\vec{r} = 1 \quad \text{if volume contains } \vec{r}\,'$$

$$\int_{\text{space}} \delta^3(\vec{r}-\vec{a})\,f(\vec{r})\,d\vec{r} = f(\vec{a})$$

<u>Curvolinear Coordinates</u>

It is not the case that

$$\delta^3(\vec{r}-\vec{r}\,') = \delta(s-s')\,\delta(\phi-\phi')\,\delta(z-z')$$

in cylindrical coordinates.

~~If~~ In curvolinear coordinates, the volume element $d\tau = (h_1\,dq_1)(h_2\,dq_2)(h_3\,dq_3)$

where q_i is some coordinate.

For example, in cylindrical

$$d\tau = (ds)(s\,d\phi)(dz)$$

$$q_1 = s, \quad q_2 = \phi, \quad q_3 = z$$

$$h_1 = 1 \quad h_2 = s \quad h_3 = 1$$

For points where q_i is uniquely defined

$$\delta^3(\vec{r}-\vec{r}\,') = \frac{1}{h_1 h_2 h_3}\,\delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

$$= \frac{1}{s}\,\delta(s-s')\,\delta(\phi-\phi')\,\delta(z-z')$$

cylindrical

$$= \frac{1}{r^2\sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')$$

spherical

At some points, multiple q_i map to the same point. For example, in cylindrical at $s=0$, all ϕ represent the same point.

At these points, the multiply define coordinate is dropped as the redundant coordinate integrated over.

$$\cancel{\delta(\vec{r}-\vec{r}\,') = \frac{1}{s\int_0^{2\pi} d\phi}\,\delta(s-s')\,\delta(z-z')}$$

$$\cancel{= \frac{1}{2\pi s}\,\delta(s-s')\,\delta(}$$

$$\overset{3}{\delta}(\vec{r}-\vec{r}\,') = \frac{1}{s\int_0^{2\pi} d\phi}\,\delta(s)\,\delta(z-z')$$

$$= \frac{1}{2\pi s}\,\delta(s)\,\delta(z-z')$$

If there are two redundant coordinates, both are integrated over. For example, at the origin in spherical coordinates

$$\overset{3}{\delta}(\vec{r}-\vec{r}\,') = \frac{1}{4\pi r^2}\,\delta(r)$$

Why worry about delta functions? They occur naturally in E+M and if we don't pay attention we lose stuff.

<u>Consider</u> $\nabla \cdot \frac{\hat{r}}{r^2}$

Using spherical coordinates

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 A_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta A_\theta \right) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\frac{\hat{r}}{r^2} = \underbrace{\frac{1}{r^2}}_{A_r} \hat{r} + \underbrace{0}_{A_\theta} \hat{\theta} + \underbrace{0}_{A_\phi} \hat{\phi}$$

$$\nabla \cdot \frac{\hat{r}}{r^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot \frac{1}{r^2} \right) = 0$$

Apply divergence thm

$$\int_V \nabla \cdot \vec{A}\, d\tau = \oint_S \vec{A} \cdot d\vec{a}$$

$$\int_V \nabla \cdot \frac{\hat{r}}{r^2}\, d\tau = 0$$

Let V be a spherical volume of radius R

so $d\vec{a} = \hat{r}\, da$, da is an area element.

$$\oint_S \frac{\hat{r}}{r^2} \cdot \hat{r}\, da = \frac{1}{R^2} \oint_S da = \frac{4\pi R^2}{R^2}$$

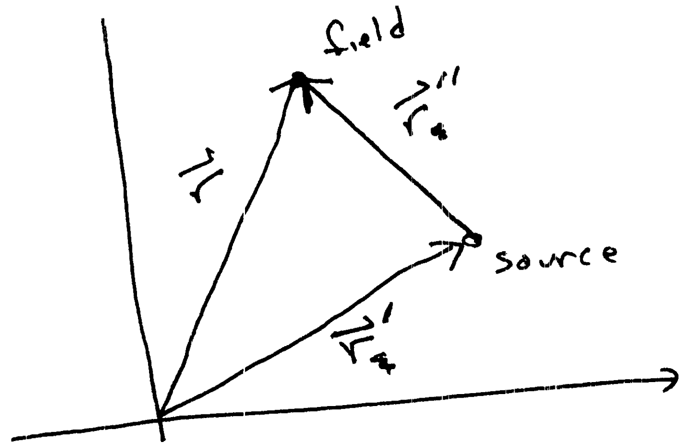
$$= 4\pi \neq 0$$

Therefore, $\nabla \cdot \frac{\hat{r}}{r^2}$ is so singular at the origin that it contributes a ~~finite~~ finite amount even though it is zero everywhere else

$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\, \delta^3(\vec{r})$$

Displacement - In general, we care about the displacement from a source point $\vec{r}'$ to a field point $\vec{r}$.

Let the displacement vector be, $\vec{r}'' = \vec{\mathscr{r}} = \vec{r} - \vec{r}'$



$$\nabla \cdot \frac{\hat{r}''}{r''^2} = 4\pi \delta^3(\vec{r}'')$$

Also, $$\nabla\left(\frac{1}{r''}\right) = -\frac{\hat{r}''}{r''^2}$$

and $$\nabla^2 \frac{1}{r''} = -4\pi \delta^3(\vec{r}'')$$