Electrostatics Review

What can we start with?

(1) Field or potential - So we already know the solution.

Find (1) Is the solution a valid ES field.

(2) Does field satisfy BC.

(3) Find other unknowns $\vec{E}$, V, σ, ρ, U, C.

(2) σ_f, ρ_f given

Find (1) E, D, V, P at point - Usually integration

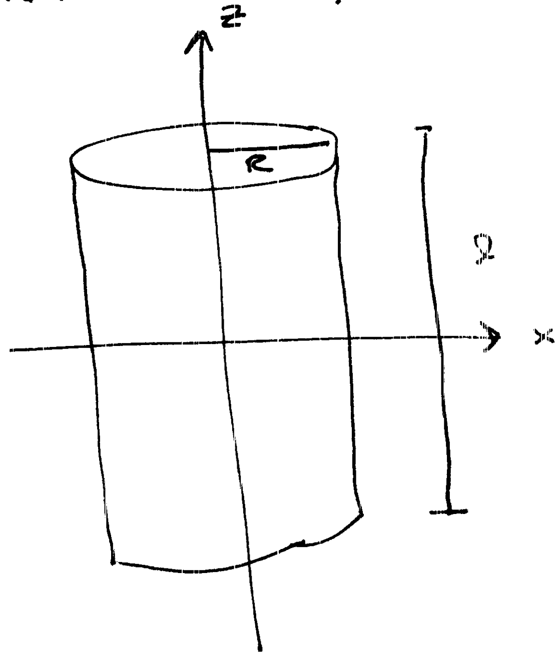
(2) E, D, V, P everywhere -
If symmetric Gauss, if not Laplace.

(3) $\vec{P}$ given

(1) Find $\vec{E}$, $\vec{D}$, V, at point - Usually by integrating bound charge

(2) Bound charge symmetric - Gauss

Ex Cylinder of length ℓ and radius R contains uniform polarization $\vec{P} = P_0 \hat{z}$. The cylinder is centered at the origin. Compute $\vec{E}, \vec{D}$ at surface. Estimate field far from cylinder along x, z axis.



The net effect of the polarization is to produce net charge densities, $\sigma_t = \hat{z} \cdot \vec{P} = P_0$ and $\sigma_b = -\hat{z} \cdot \vec{P} = -P_0$. Once we have these bound charges, we can compute the fields ignoring the polarization.

First, lets get the far fields. The dipole moment is $\vec{P}_{dip} = \ell Q = \ell \pi R^2 \sigma_t = \ell \pi R^2 P_0$

As $z \to \infty$, $\vec{E}(0,0,z) = \dfrac{2kp}{z^3}\hat{z} = \dfrac{\ell \pi R^2 P_0}{2\pi \varepsilon_0 z^3}\hat{z}$

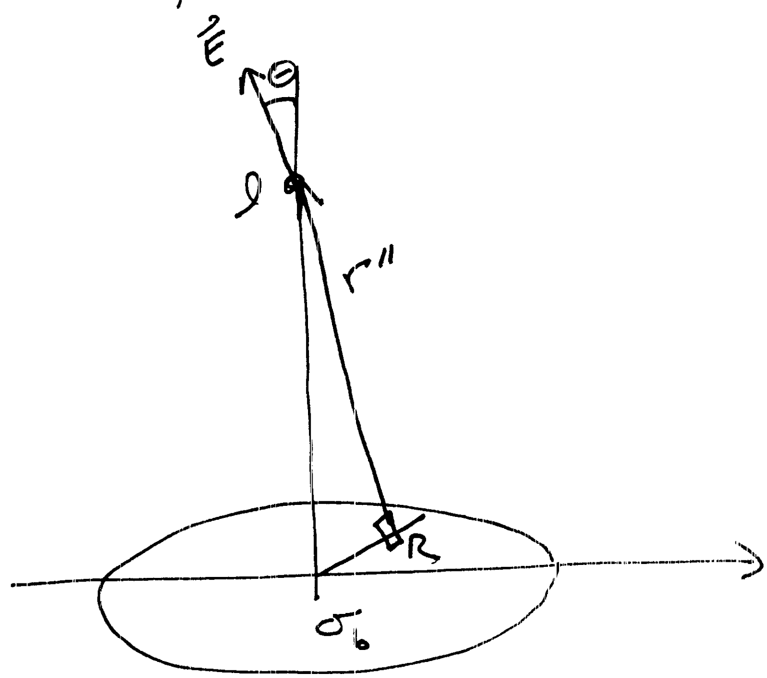
$$= \frac{\ell R^2 P_0}{2 \varepsilon_0 z^3}\hat{z}$$

And as $x \to \infty$

$$E(x,0,0) = -\frac{kp}{x^3}\hat{z}$$

$$= \frac{-9\pi R^2 P_0}{4\pi\epsilon_0 x^3}\hat{z} = -\frac{9R^2P_0}{4\epsilon_0 x^3}\hat{z}$$

The surface field is the sum of the field of σ_t and σ_b. The field of σ_t is zero by symmetry. So we are left with the field of a disk of charge with charge density $\sigma_b = -P_0$.



Method I

$$\vec{r}_p = (0, 0, z) \qquad \vec{r}\,' = (x, y, 0)$$

$$= s'\hat{s}$$

$$\vec{r}\,'' = \vec{r}_p - \vec{r}\,' = z\hat{z} - s'\hat{s}'$$

$$r'' = \sqrt{z^2 + s'^2}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r''^2}\, \hat{r}'' = \frac{1}{4\pi\epsilon_0} \int \frac{dq\, \vec{r}\,''}{r''^3}$$

$$= \frac{1}{4\pi\epsilon_0} \int \frac{dq}{(z^2+s'^2)^{3/2}} \cdot (z\hat{z} - s'\hat{s}')$$

The s' integration is zero by symmetry.

$$dq = \sigma_b\, ds'\, s'\, d\phi'$$

$$\vec{E} = \frac{z\hat{z}}{4\pi\epsilon_0} \int \frac{dq}{(z^2+s'^2)^{3/2}} =$$

$$= \frac{z\sigma_b\hat{z}}{4\pi\epsilon_0} \int_0^R ds' \int_0^{2\pi} d\phi' \frac{s'}{(z^2+s'^2)^{3/2}}$$

$$= \frac{z\sigma_b\hat{z}}{2\epsilon_0} \int_0^R \frac{ds'\, s'}{(z^2+s'^2)^{3/2}}$$

$$\vec{E} = \frac{l\,\sigma_b}{2\varepsilon_0}\hat{z}\left(\frac{1}{l} - \frac{1}{\sqrt{l^2+R^2}}\right)$$

$$= -\frac{P_0}{2\varepsilon_0}\hat{z} + \frac{P_0}{2\varepsilon_0}\frac{l}{\sqrt{l^2+R^2}}\hat{z}$$

Suppose $l = R$

$$\vec{E} = -\frac{P_0}{2\varepsilon_0}\hat{z}\underbrace{\left(1 - \frac{1}{\sqrt{2}}\right)}_{0.29}$$

Compute $\vec{D}$ ~~at surface~~ Outside $\vec{P} = 0$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

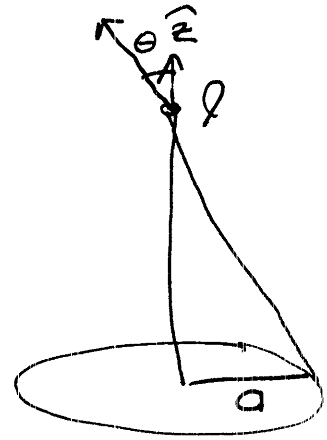
~~$= \frac{P_0}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$~~

$$= -\frac{P_0}{2}\hat{z} + \frac{P_0}{2}\frac{l}{\sqrt{l^2+R^2}}\hat{z}$$

Method 2

Use previous intermediate result

Field of Ring Radius a Charge Density λ



$$\vec{E}(\ell) = \frac{k q}{r''^2} \cos\theta \, \hat{z} = \frac{k(2\pi a \lambda)}{(\sqrt{a^2+\ell^2})^2} \cdot \frac{\ell}{\sqrt{\ell^2+a^2}}$$

$$= \frac{a \lambda \ell}{2\varepsilon_0 (\ell^2+a^2)^{3/2}} \hat{z}$$

Now divide the disk up into rings of radius s and charge density $\sigma \, ds = d\lambda$

$$\vec{E} = \hat{z} \int_0^R \frac{s \, d\lambda \, \ell}{2\varepsilon_0 (\ell^2+s^2)^{3/2}} = \frac{\hat{z} \, \ell \sigma}{2\varepsilon_0} \int_0^R \frac{s \, ds}{(\ell^2+s^2)^{3/2}}$$

$$= \hat{z} \frac{\ell \sigma}{2\varepsilon_0} \left(\frac{1}{\ell} - \frac{\ell}{\sqrt{\ell^2+R^2}} \right)$$

Could we attack this with another method?

The problem is azimuthally symmetric; we could try solving Laplace's eqn and fitting the charge densities to the boundaries. This would give the full solution, but would be very difficult.

We could also try it with $\vec{D}$,

$$\nabla \cdot \vec{D} = \rho_f = 0$$

but this just tells us $\vec{D}$ is divergence free except at the boundaries. Not much help. I think much of the problem people are having with $\vec{D}$ is when it can be used successfully.

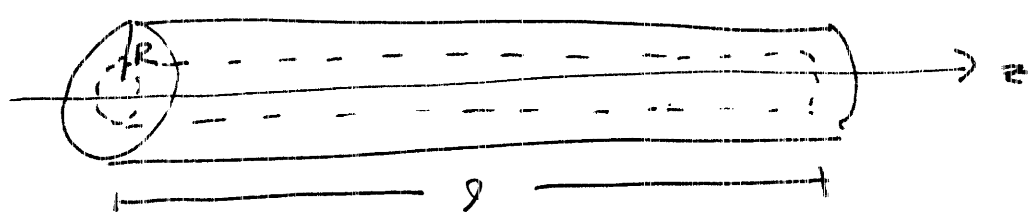
(1) $\nabla \cdot \vec{D} = \rho_f$ is true always.

(2) It is useful when symmetry allows Gaussian techniques -or- as a method to apply BC in a boundary value solution.

Ex A non-uniform volume charge $\rho(s) = \rho_0 \frac{e^{-s/a}}{s}$ filled the cylindrical region $s < R$. The volume charge is embedded in a linear dielectric with dielectric constant $K = \varepsilon_r$. Find Everything.

Sln

Note, $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ but the ρ that was given is ρ_f, $\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0}(\rho_f + \rho_b)$



Apply Gauss' Law for Displacement to a cylindrical surface of radius s s.t. $s < R$

$$\nabla \cdot \vec{D} = \rho_f$$

$$\Phi_d = \int_S \vec{D} \cdot \hat{n}\, da = Q_{fenc} = 2\pi s l D$$

$$Q_{fenc} = l \int \rho\, da \qquad da = ds\, s\, d\phi$$

$$= l \int_0^s ds \int_0^{2\pi} d\phi\, s\, \rho = l \rho_0 \int_0^s ds \int_0^{2\pi} d\phi\, e^{-s/a}$$

$$Q_{fenc} = 2\pi l \rho_o \int_0^s e^{-s/a}\, ds$$

$$= -2\pi l a \rho_o \, e^{-s/a} \Big|_0^s$$

$$= 2\pi l a \rho_o \left(1 - e^{-s/a}\right)$$ Note, dimensionally correct.

$$\Phi_d = 2\pi s l D = 2\pi l a \rho_o \left(1 - e^{-s/a}\right)$$

$$\vec{D}_i = \rho_o \frac{a}{s}\left(1 - e^{-s/a}\right)\hat{s}$$

Since linear, $\vec{D}_i = \varepsilon \vec{E}_i = \varepsilon_o \varepsilon_r \vec{E}_i = \varepsilon_o (1+\chi_e)\vec{E}_i$

$$\vec{E}_i = \frac{\rho_o}{\varepsilon_o \varepsilon_r} \frac{a}{s}\left(1 - e^{-s/a}\right)\hat{s}$$

<u>Polarization</u>

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

$$\vec{P} = \frac{\rho_o \chi_e}{\varepsilon_r} \frac{a}{s}\left(1 - e^{-s/a}\right)\hat{s}$$

$$= \rho_o \left(\frac{\chi_e}{1+\chi_e}\right) \frac{a}{s}\left(1 - e^{-s/a}\right)\hat{s}$$

Field Outside Cylinder $s > R$

$$Q_{fenc} = 2\pi l a \rho_o \left(1 - e^{-R/a}\right)$$

$$2\pi s l D_o = Q_{fenc} = 2\pi l a \rho_o \left(1 - e^{-R/a}\right)$$

$$\vec{D}_o = \rho_o \left(\frac{a}{s}\right)\left(1 - e^{-R/a}\right)\hat{s}$$

$$\vec{E}_o = \frac{\vec{D}_o}{\varepsilon} = \frac{\vec{D}_o}{\varepsilon_o} = \frac{\rho_o}{\varepsilon_o}\left(\frac{a}{s}\right)\left(1 - e^{-R/a}\right)\hat{s}$$

Bound Charge Densities

Volume Charge Density

$$\rho_b = -\nabla \cdot \vec{P}$$

$$-\rho_b = \frac{1}{s}\frac{\partial}{\partial s}(s P_s) + \underbrace{\frac{1}{s}\frac{\partial P_\phi}{\partial \phi}}_{0} + \underbrace{\frac{\partial P_z}{\partial z}}_{0}$$

$$\rho_b = -\frac{1}{s}\left[\frac{\partial}{\partial s}\frac{\rho_o \chi_e a}{\varepsilon_r}\left(1 - e^{-s/a}\right)\right]$$

$$= -\frac{1}{s}\left(\frac{\rho_o \chi_e a}{\varepsilon_r}\right)\left(\frac{1}{a}\right)e^{-s/a}$$

$$= -\frac{1}{s}\frac{\rho_o \chi_e}{\varepsilon_r}e^{-s/a}$$

Which checks because in a linear dielectric

$$\rho_b = -\frac{\chi_e}{\varepsilon_r}\rho_f$$

<u>Bound Surface Charge</u>

$$\sigma_b = \vec{P}\cdot\hat{s}$$

$$= \frac{\rho_0 \chi_e}{\varepsilon_r}\frac{R}{s}\left(1 - e^{-R/a}\right)$$

Suppose we are not working with a linear medium, how much of the above goes forward

We still find

$$\vec{D}_i = \rho_0 \frac{a}{s}\left(1 - e^{-s/a}\right)\hat{s}$$

$$\vec{D}_o = \rho_0\left(\frac{a}{s}\right)\left(1 - e^{-R/a}\right)\hat{s}$$

$$\vec{E}_o = \frac{\rho_0}{\varepsilon_0}\left(\frac{a}{s}\right)\left(1 - e^{-R/a}\right)\hat{s}$$

We cannot find $\vec{E}_i$, $\vec{P}$, ρ_b, or σ_b because we are missing physical information about the response of the material.

Ex Suppose we know the response, but it's not linear.

$$\vec{P} = \varepsilon_0 \chi_e (1 + \alpha E_i) \vec{E_i}$$

Once again, $\vec{D_i}$, $\vec{D_o}$, and $\vec{E_o}$ are unchanged.

$$\vec{D_i} = \varepsilon_0 \vec{E_i} + \vec{P}$$

$$= \varepsilon_0 \vec{E_i} + \varepsilon_0 \chi_e (1 + \alpha E_i) \vec{E_i}$$

$$\vec{D_i} = (\varepsilon + \varepsilon_0 \chi_e \alpha E_i) \vec{E_i}$$

Substitute $\vec{D_i}$

$$|\vec{D_i}| = \rho_0 \frac{a}{s} (1 - e^{-s/a}) = \varepsilon E_i + \varepsilon_0 \chi_e \alpha E_i^2$$

⇒ Use quadratic formula to find $E_i(s)$

⇒ From $E_i(s)$, write polarization $P(s)$

⇒ From polarization, find ρ_f, σ_f

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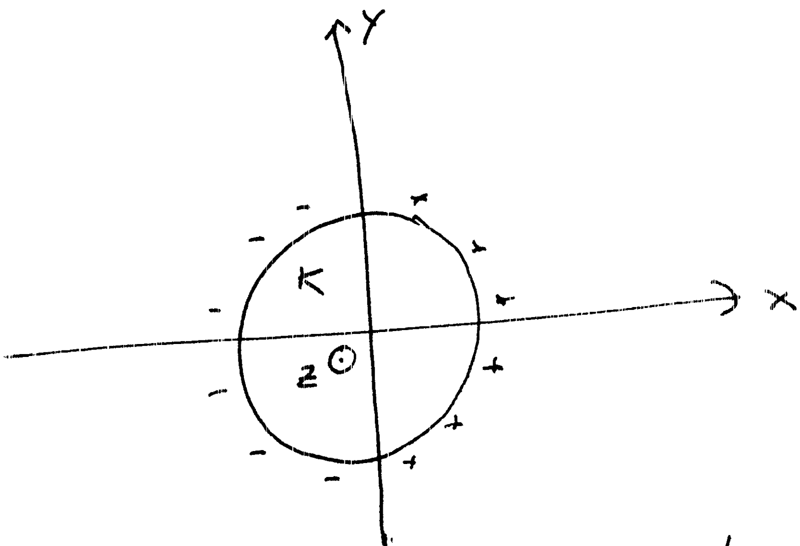
<u>Ex</u> - Infinite Dielectric Cylinder with surface charge density $\sigma = \sigma_0 \cos\phi$.



<u>S/n</u> No symmetry, so Gauss is out. Dielectric so can't just integrate to get field or potential.

Solve Laplace's Eqn in Cylindrical Symmetry

$$V = \sum_n A_n s^n \cos n\phi + B_n s^n \sin n\phi + C_n s^{-n} \cos n\phi + D_n s^{-n} \sin n\phi$$

$$\int_0^{2\pi} d\phi \sin n\phi \sin m\phi = \pi \delta_{nm}$$

Boundary Conditions

$$V_i = V_o \quad \text{at} \quad s = a$$

$$\varepsilon_{out} \left.\frac{\partial V_o}{\partial s}\right|_a - \varepsilon_{in} \left.\frac{\partial V_i}{\partial s}\right|_a = -\sigma_f$$

Solution

Inside

$$V_i = \sum_n A_n s^n \cos n\phi + B_n s^n \sin n\phi$$

Outside

$$V_o = \sum_n C_n s^{-n} \cos n\phi + D_n s^{-n} \sin n\phi$$

Discarding terms that go to ∞.

BC $\quad V_i(a,\phi) = V_o(a,\phi)$

$$\sum_n A_n a^n \cos n\phi + B_n a^n \sin n\phi$$

$$= \sum_n C_n a^{-n} \cos n\phi + D_n a^{-n} \sin n\phi$$

Use orthogonality to match terms

$$A_n a^n = C_n a^{-n}$$

$$C_n = a^{2n} A_n$$

$$B_n a^n = D_n a^{-n}$$

$$D_n = a^{2n} B_n$$

Other BC (Electrostatic)

Need

$$\left.\frac{\partial V_o}{\partial s}\right|_a = \sum_n -n \left(a^{-(n+1)} \cos n\phi \, C_n + a^{-(n+1)} D_n \sin n\phi \right)$$

$$= \sum_n -n a^{n-1} \left(A_n \cos n\phi + B_n \sin n\phi \right)$$

$$\left.\frac{\partial V_i}{\partial s}\right|_a = \sum_n n a^{n-1} \left(A_n \cos n\phi + B_n \sin n\phi \right)$$

$$\varepsilon_0 \left.\frac{\partial V_o}{\partial s}\right|_a - \varepsilon_0 \varepsilon_r \left.\frac{\partial V_i}{\partial s}\right|_a = -\sigma_f = -\sigma_0 \cos\phi$$

Substitute

$$\sum_n -n a^{n-1} \left(A_n \cos n\phi + B_n \sin n\phi \right)$$

$$- \varepsilon_r \sum n a^{n-1} \left(A_n \cos n\phi + B_n \sin n\phi \right)$$

$$= - \frac{\sigma_0}{\varepsilon_0} \cos\phi$$

$$\sum_n n(\varepsilon_r + 1) a^{n-1} \sum \left(A_n \cos n\phi + B_n \sin \phi \right)$$

$$= \frac{\sigma_0}{\varepsilon_0} \cos\phi$$

Use orthogonality relation to assert

$$A_n = B_n = 0 \quad \text{if } n \neq 1$$

$$B_1 = 0$$

$$(1)(\varepsilon_r + 1) a^{1-1} A_1 \cos\phi = \frac{\sigma}{\varepsilon_0} \cos\phi$$

$$A_1 = \frac{\sigma}{\varepsilon_0(\varepsilon_r + 1)}$$

Potentials

$$V_i = A_1 s \cos\phi$$

$$= \frac{\sigma}{\varepsilon_0(\varepsilon_r+1)} s \cos\phi$$

$$V_o = C_1 s^{-1} \cos\phi$$

$$C_1 = a^2 A_1$$

$$V_o = \frac{\sigma}{\varepsilon_0(\varepsilon_r+1)} \frac{a^2}{s} \cos\phi$$

Fields

Inside $\quad V_i = \dfrac{\sigma}{\varepsilon_0(\varepsilon_r+1)} x$

$$\vec{E}_i = -\nabla V_i = \frac{-\sigma}{\varepsilon_0(\varepsilon_r+1)} \hat{x}$$

$$\vec{E}_0 = -\nabla V_0 = -\left[\frac{\partial V_0}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial V_0}{\partial \phi}\hat{\phi} \right]$$

$$= \frac{\sigma}{\epsilon_0(\epsilon_r+1)} \frac{a^2}{s^2} \left[\cos\phi\, \hat{s} + \sin\phi\, \hat{\phi} \right]$$

$$[\;] = \cos\phi\left(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right)$$

$$+ \sin\phi\left(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right)$$

$$= \left(\cos^2\phi - \sin^2\phi\right)\hat{x} + 2\cos\phi\sin\phi\,\hat{y}$$

$$= \cos 2\phi\,\hat{x} + \sin 2\phi\,\hat{y}$$