Displacement

The bound charge that results from a polarization $\vec{P}$ is just charge, so

$$\nabla \cdot \vec{E} = \rho_b / \varepsilon_0$$

$$\nabla \times \vec{E} = 0$$

$$\vec{E} = \int_V \frac{k \rho_b(\vec{r}') \, d\tau' \, \hat{r}''}{r''^2}$$

If there are additional charges floating around not produced by polarization, they just get added in

$$\nabla \cdot \vec{E} = \frac{\rho_b}{\varepsilon_0} + \frac{\rho_f}{\varepsilon_0}$$

where I have labeled the other charge ρ_f.

We will refer to the other charge as <u>free charge</u> because charge not resulting from polarization doesn't have a ring to it.

Work on Gauss' Law

$$\varepsilon_0 \nabla\cdot\vec{E} = \rho_f + \rho_b$$

$$\varepsilon_0 \nabla\cdot\vec{E} = \rho_f - \nabla\cdot\vec{P}$$

$$\nabla\cdot\left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho_f$$

Dfn - Electric Displacement

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\nabla\cdot\vec{D} = \rho_f$$

Integral Form

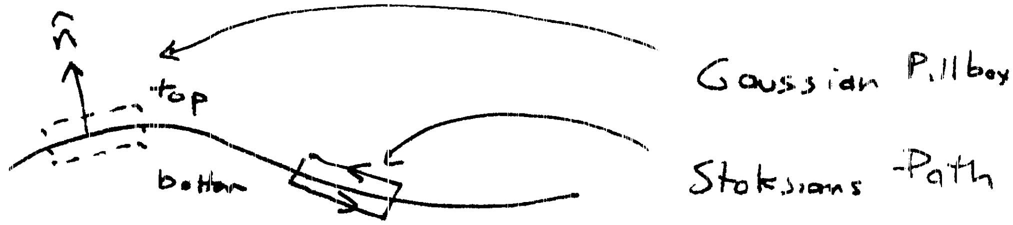
$$\int_V \nabla\cdot\vec{D}\, d\tau = \int_V \rho_f\, d\tau = \oint_S \vec{D}\cdot\hat{n}\, da$$

divergence thm

$$\int_V \rho_f\, d\tau = Q_{f\,enc}$$

The free charge enclosed in the Gaussian surface.

Electrostatic Boundary Conditions



$$\Phi_d = \vec{D}_{top}\cdot\hat{n}\,A - \vec{D}_{bottom}\,\hat{n}\,A = \sigma_f$$

$$D^{\perp}_{top} - D^{\perp}_{bottom} = \sigma_f$$

$D^{\perp}$ = component of $\vec{D}$ $\perp$ to surface.

$$\oint_C \vec{E}\cdot d\vec{l} = \int_S \nabla\times\vec{E}\cdot d\vec{a} = 0$$

$$= \int_S \left(\nabla\times\frac{\vec{D}}{\varepsilon_0} - \nabla\times\frac{\vec{P}}{\varepsilon_0}\right)\cdot d\vec{a}$$

$$\oint_C \vec{D}\cdot d\vec{l} = \int_C \vec{P}\cdot d\vec{l}$$

$$D^{\parallel}_{top} - D^{\parallel}_{bottom} = P^{\parallel}_{top} - P^{\parallel}_{bottom}$$

where $D^{\parallel}$ is the component of D parallel to the surface

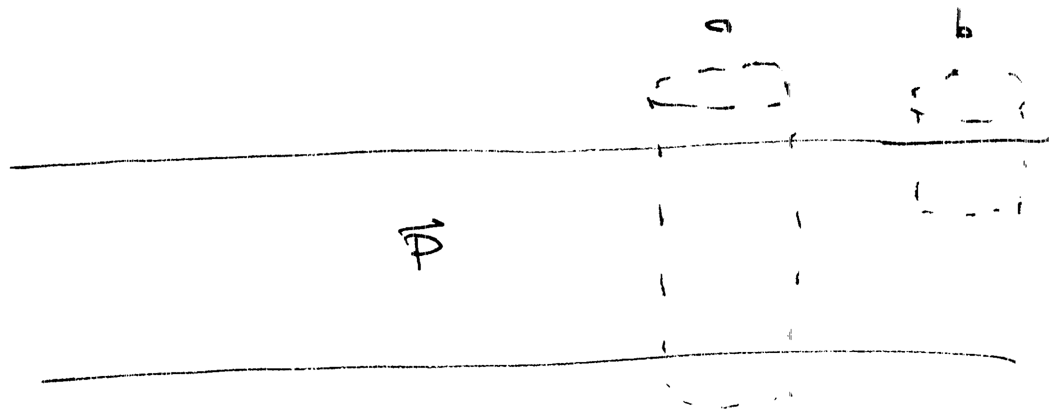
Don't get carried away

$$\nabla \times \vec{D} = \varepsilon_0 \nabla \times \vec{E} + \nabla \times \vec{P} = \nabla \times \vec{P} \neq 0$$

always

$$\text{so} \quad \vec{D} \neq \int \frac{\rho_f \, d\tau' \, \hat{r}''}{r''^2}$$

Ex Back to the slab with uniform polarization $\vec{P}$



There is no free charge anywhere $\rho_f = 0$

$$\nabla \cdot \vec{D} = 0 \qquad \Rightarrow \vec{D} = 0$$

or at least $\vec{D} =$ constant.

Gaussian surface a has $Q_{f\,enc} = 0$ and reflection symmetry $\vec{D} = 0$ above and below the slab. Gaussian surface b encloses zero net charge, so $\vec{D} = 0$ inside the slab.

So $\vec{D} = 0$.

Inside

$$\vec{D} = 0 = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{E} = \frac{-\vec{P}}{\varepsilon_0}$$

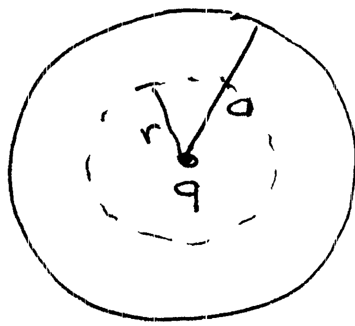
Just what we found earlier.

Outside $\vec{P} = 0$

$$\vec{D} = 0 = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} \Rightarrow \vec{E} = 0$$

Now suppose the polarization results from putting some material in a field. We have no way to know what the polarization is and it may change in a complicated manner from point to point, but what we did earlier is general.

Ex Point charge embedded in dielectric material of radius R. Compute $\vec{D}$ everywhere, $\vec{E}$ when you can.



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Displacement Flux $\quad \Phi_d = \int_S \vec{D} \cdot \hat{n}\, dA = Q_{fenc}$

where S is a spherical Gaussian surface of radius r.

Region I $\quad r < a \quad Q_{fenc} = q$

$$\Phi_d = 4\pi r^2 D = q$$

$$\vec{D} = \frac{q}{4\pi r^2}\hat{r}$$

we cannot calculate $\vec{E}$ since we don't know $\vec{P}$.

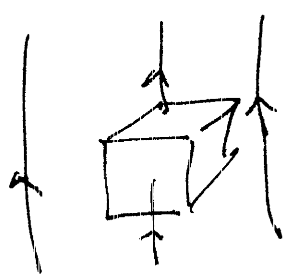
Region II $\quad r > a \quad Q_{encf} = q$ also

$$\vec{D} = \frac{q}{4\pi r^2}\,\hat{r}$$

but now $\vec{P} = 0$, so

$$\vec{D} = \varepsilon_0 \vec{E} \implies \vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{r}$$

If we know $\vec{E}$ inside the dielectric, can we calculate $\vec{P}$ $\Rightarrow$ Generally, no.



If we apply a field to a small block of dielectric, we change its atomic energy environment, and its thermodynamic free energy. All hell can break loose; we could fore a phase transition.

In general,

$$\vec{P} = \overleftrightarrow{\chi}_e \vec{E}$$

susceptibility tensor

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

where there is nothing to prevent χ_{ij} from being a function of the field strength. For many materials, like crystals χ_{ij} are independent of E_i for normal strength fields.

For some materials, called <u>linear dielectrics</u> the polarization points in the same direction as the field.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

<u>Electric Susceptibility</u> (χ_e)

For these materials,

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{P}$$

$$= \varepsilon_0 (1 + \chi_e) \vec{E}$$

<u>Dfn</u> <u>Permittivity</u>

$$\varepsilon = \varepsilon_0 (1 + \chi_e)$$

<u>Dfn</u> <u>Relative Permittivity</u>

$$\varepsilon_r = 1 + \chi_e$$

<u>Dfn</u> Dielectric Constant $K = \varepsilon_r = 1 + \chi_e$

<u>So</u> $\vec{D} = \varepsilon \vec{E}$ for linear dielectric

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Return to point charge in dielectric sphere. If sphere is linear dielectric

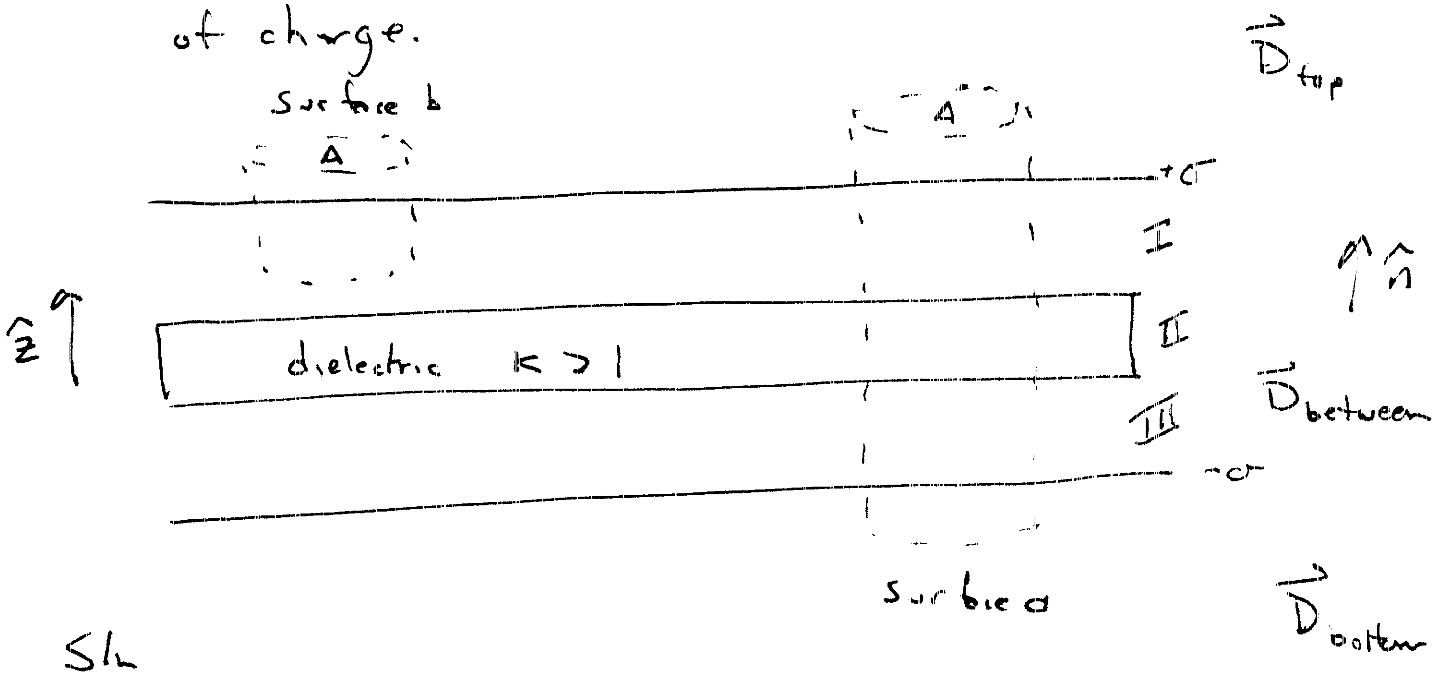
$$\vec{D} = \frac{q}{4\pi r^2} \hat{r} = \varepsilon \vec{E}$$

$$\cancel{\vec{E} = \frac{q}{4\pi\varepsilon r^2} \hat{r}}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon r^2}\hat{r} = \frac{q}{4\pi\varepsilon_r\varepsilon_0 r^2}\hat{r}$$

$$= \frac{q}{4\pi\varepsilon_0 \kappa r^2}\hat{r}$$

Ex Dielectric between two equal, but opposite planes of charge.



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Surface a $\quad Q_{free} = 0$, outer fields equal, opposite by reflection

$$\Phi_d = \vec{D}_{top} \cdot \hat{n} A - \vec{D}_{bottom} \cdot \hat{n} A = Q_{free} = 0$$

$$\vec{D}_{top} = -\vec{D}_{bottom}$$

$$\vec{D}_{top} = \vec{D}_{bottom} = 0$$

Surface b $\quad Q_{fenc} = \sigma A$

$$\Phi_d = \vec{D}_{top} \cdot \hat{n} A - \vec{D}_{between} \cdot \hat{n} A = Q_{fenc} = \sigma A$$

$$\vec{D}_{between} = -\sigma \hat{z}$$

Region I $\quad$ Above the top plane

$$\vec{D}_{top} = 0 \qquad \vec{P}_{top} = 0 \qquad \vec{E}_{top} = 0$$

Region II $\quad$ Between the planes, but outside of the dielectric

$$\vec{D}_{between} = -\sigma \hat{z} = +\varepsilon_0 \vec{E}, \qquad \vec{P}_{between} = 0$$

$$\vec{E}_{between} = \frac{-\sigma}{\varepsilon_0} \hat{z}$$

Region III

$$\vec{D}_{diel} = -\sigma \hat{z} = \varepsilon \vec{E}$$

$$\vec{E}_{diel} = \frac{-\sigma}{\varepsilon} \hat{z} = \frac{-\sigma}{K \varepsilon_0} \hat{z} = \frac{-\sigma}{\varepsilon_0 (1 + \chi_e)} \hat{z}$$

$$\vec{P}_{diel} = \chi_e \varepsilon_0 \vec{E}$$

$$= -\sigma \left(\frac{\chi_e}{1 + \chi_e} \right) \hat{z}$$

Region IV Below the bottom plane

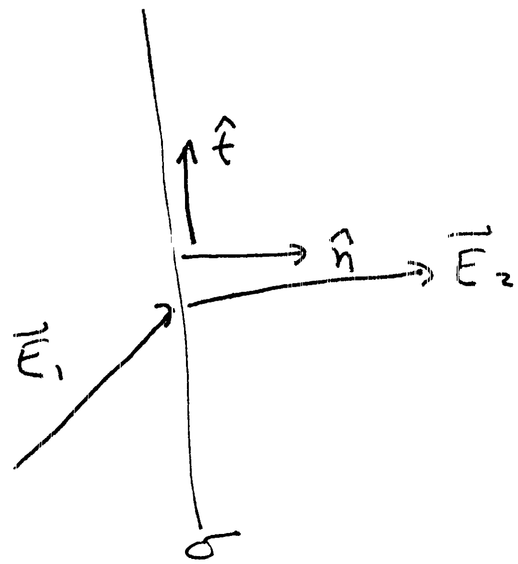
$$\vec{D}_{bottom} = 0 \qquad \vec{P}_{bottom} = 0 \qquad \vec{E}_{bottom} = 0$$

Charge Densities

$$\rho_b = -\nabla \cdot \vec{P} = 0$$

$$\sigma_{b,top} = \vec{P} \cdot \hat{n} = \frac{-\chi_e}{1+\chi_e}\,\sigma$$

Back to Boundary Conditions

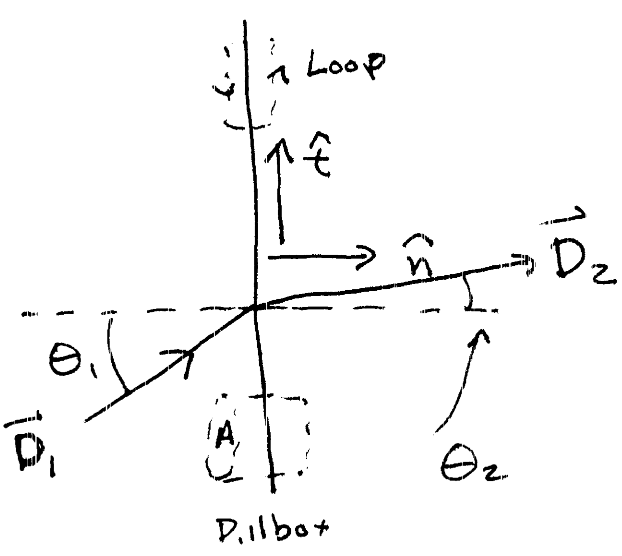


Without Dielectrics

$$\vec{E}_2 \cdot \hat{n} - \vec{E}_1 \cdot \hat{n} = \sigma/\varepsilon_0$$

$$\vec{E}_2 \cdot \hat{t} = \vec{E}_1 \cdot \hat{t}$$

Same Problem with Dielectrics



Gaussian Pill box

$$\frac{\Phi_d}{A} = \vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = \sigma_f$$

If $\sigma_f = 0$, $D_2 \cos\theta = D_1 \cos\theta$

Stokesian Loop

$$\oint \vec{E} \cdot d\vec{s} = 0 \qquad \vec{E}_1 \cdot \hat{t} = \vec{E}_2 \cdot \hat{t}$$

$$E_1 \sin\theta = E_2 \sin\theta$$

$$\frac{E_1}{D_1} \tan\theta_1 = \frac{E_1}{D_1} \tan\theta_2$$

$$D_1 = \varepsilon_0 \varepsilon_{r1} E_1$$

$$\frac{1}{\varepsilon_{r1}} \tan\theta_1 = \frac{1}{\varepsilon_{r2}} \tan\theta_2$$

$\Longrightarrow$ $\vec{D}$ bends at dielectric interface

A few points

(1) In a linear dielectric, since $\vec{D} = \varepsilon \vec{E}$ the curl of $\vec{D}$ is zero. However, the curl of $\vec{D}$ is not zero at any interfaces.

If, however, the dielectric material completely fills the field space (anywhere field exists), there are no interfaces and $\nabla \cdot \vec{D} = \rho_f$ and $\nabla \times \vec{D} = 0$.

These are the same equations we would have if the dielectric vanished, so we can calculate the field without the dielectric $\vec{E}_0$ and $\vec{D}$ becomes

$$\vec{D} = \varepsilon_0 \vec{E}_0$$

Therefore $$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\vec{E}_0}{\varepsilon_r} = \frac{\vec{E}_0}{K}$$

$\Rightarrow$ The field is reduced by K

(2) The $\vec{D}$ field has nothing to do with the potential difference

$$\Delta V = -\int \vec{E} \cdot d\vec{l}$$

period.

(3) The free and bound charge are related.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

$$\rho_b = -\nabla \cdot \vec{P} = -\varepsilon_0 \chi_e \nabla \cdot \vec{E}$$

$$\cancel{-\varepsilon_0 \chi_e}$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\rho_b = -\varepsilon_0 \chi_e \nabla \cdot \left(\frac{\vec{D}}{\varepsilon} \right)$$

$$= -\frac{\chi_e}{\varepsilon_r} \nabla \cdot \vec{D} = -\frac{\chi_e}{\varepsilon_r} \rho_f$$

$\Rightarrow$ In regions without ρ_f, there is no ρ_b.

(4) In ~~a~~ free charge free regions,

$$\rho_f = \rho_b = 0$$

without dielectric interfaces

$$\nabla \cdot \vec{D} = 0 = \varepsilon \nabla \cdot \vec{E} = 0$$

and $\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla V$

(5) <u>Potential Boundary Conditions</u>

$$\vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = \sigma_f$$

$$\varepsilon_2 \vec{E}_2 \cdot \hat{n} - \varepsilon_1 \vec{E}_1 \cdot \hat{n} = \sigma_f$$

(A) $$\varepsilon_2 \frac{\partial V_2}{\partial n} - \varepsilon_1 \frac{\partial V_1}{\partial n} = -\sigma_f$$

(B) $$V_2 = V_1 \quad \text{(always)}$$