Electromotive Force

Consider the work done by the electromagnetic fields as a charge q is moved along a path C

$$\text{Work} = \int_C (\vec{F}_e + \vec{F}_m) \cdot d\vec{l}$$

$$= q\int_C (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$$

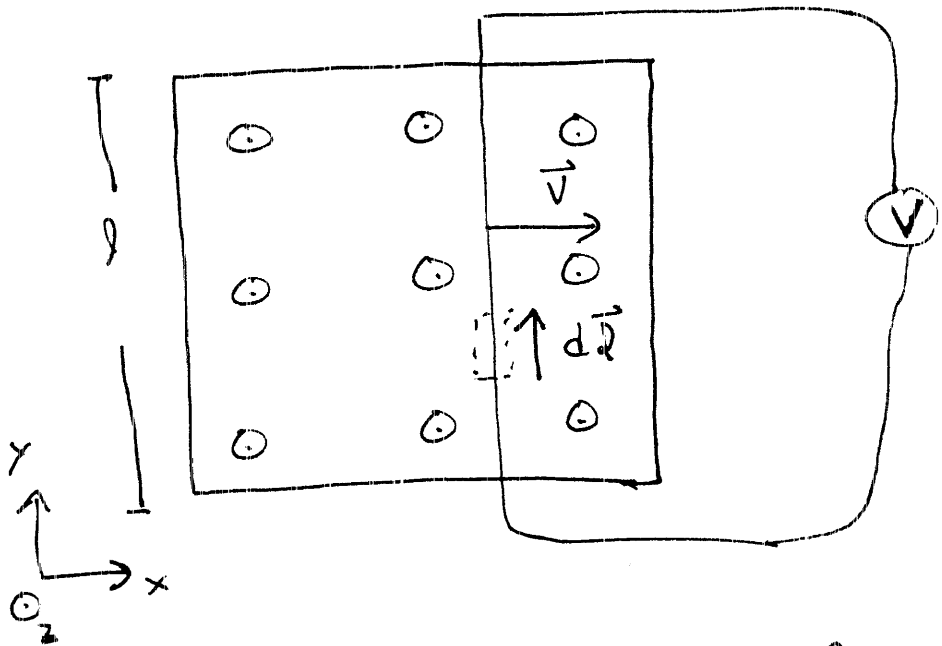
<u>Electromotive Force</u> ($\mathcal{E}$ or emf) - Work per unit charge done by electromagnetic fields to move a charge through a path C

$$\text{emf} = \frac{W}{q} = \int_C (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$$

<u>Motional EMF</u> The part of the emf resulting from the motion of C

$$\text{emf} = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

Ex Pull wire out of magnet



$$\vec{B} = B_0 \hat{z} \qquad \vec{v} = v_0 \hat{x} \qquad d\vec{l} = dy \, \hat{y}$$

$$\vec{v} \times \vec{B} = v_0 B_0 (\hat{x} \times \hat{z}) = -\hat{y} v_0 B_0$$

$$(\vec{v} \times \vec{B}) \cdot d\vec{l} = -v_0 B_0 \, dy \, (\hat{y} \cdot \hat{y}) = -v_0 B_0 \, dy$$

Motional EMF

$$\text{emf} = \int_c \vec{v} \times \vec{B} \cdot d\vec{l} = -v_0 B_0 l$$

$\Rightarrow$ We would measure emf in volts on a voltmeter.

Note, the directions are crucial.

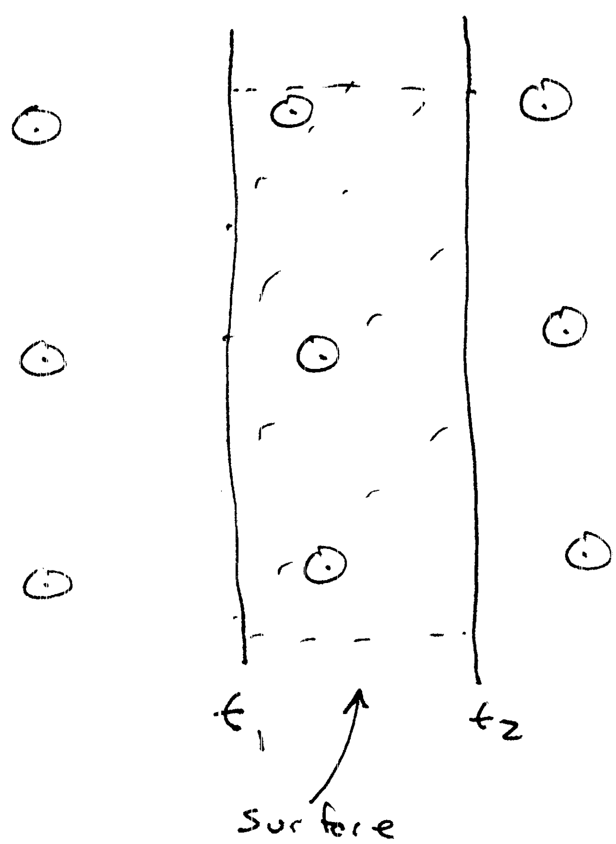
(1) If the wire was pulled out of the page

$\vec{v} = v_0 \hat{z} \Rightarrow \vec{v} \times \vec{B} = 0$

(2) If the wire was pulled to the top of the page, $\vec{v} = v_0 \hat{y} \Rightarrow \vec{v} \times \vec{B} = v_0 B_0 \hat{x}$

$\vec{v} \times \vec{B} \cdot d\vec{l} = 0$

$\Rightarrow$ The key to having a non-zero emf is for the surface formed by the trajectory of the path to have a non-zero magnetic flux through the surface.



$$\frac{d\Phi}{dt} = Blv$$

<u>Magnetic Flux</u> (Φ_m) The magnetic flux through surface S is

$$\Phi_m = \int_S \vec{B} \cdot d\vec{a}$$

$\Rightarrow$ If the field is uniform,

$$\Phi_m = \vec{B} \cdot \hat{n}\, A = BA\cos\theta$$

For our wire moving in the field at constant velocity, the flux through the surface traced out between t_1 and t_2 is

$$\Phi_m = B_0 \ell\, v_0 (t_2 - t_1)$$

The emf then is the time rate of change of this flux

we'll get to this

$$\text{emf} = -\frac{d\Phi_m}{dt}$$

Ex How many volts? Suppose $B_0 = \frac{1}{4}$ T,

$v_0 = 10$ m/s , $l = 10$ cm.

$$\text{emf} = B_0 v_0 l = \frac{1}{4} \text{ V}$$

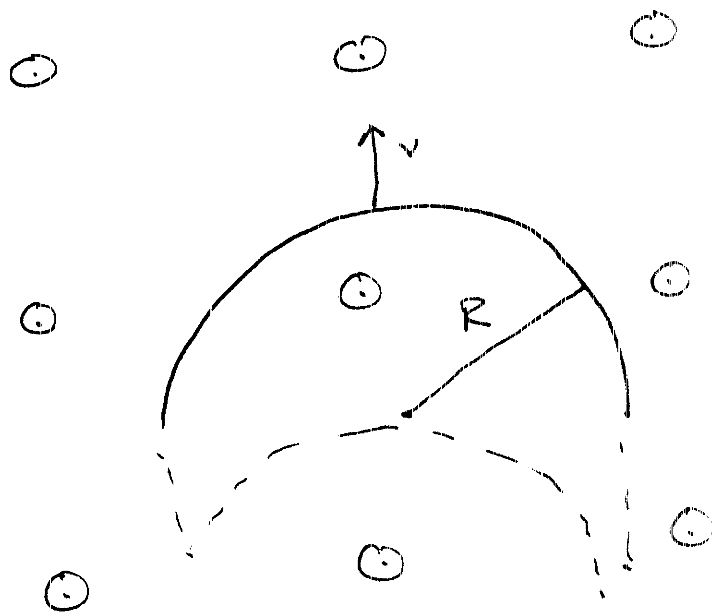
Multiple Turns (N) - We will often use surfaces that are wound with multiple turns of wire, the flux through such a surface is N times the flux through 1 turn.

$$\Phi_m = N \oint \vec{B} \cdot d\vec{a}$$

Flux Rule The emf around a closed curve C (or a curve that can be completed) is proportional to the change in flux through the surface bounded by C

$$\text{emf} = -\frac{d\Phi_m}{dt}$$

Ex The flux rule is quite a bit more powerful than the motional emf integral. Compute emf across half-circle of radius R moving in field B_0.



Note $\frac{\text{Magnetic force}}{q}$ is a constant $\vec{v} \times \vec{B} = v_0 B_0 \hat{x}$

but $\vec{v} \times \vec{B}$ is not parallel to $d\vec{l}$ so the integration is a pain.

Flux Rule $\frac{d\Phi_m}{dt} = -B_0 2Rv = \text{emf}$

Now consider changing our moving rod system by moving the magnet instead of the rod. Relativity says we must read the same thing on the meter

but $\vec{v} = 0$.

$$\text{emf} = \int (\vec{E} + \underbrace{\vec{v} \times \vec{B}}_{0}) \cdot d\vec{l}$$

$\Rightarrow \vec{E} \neq 0$

$\Rightarrow$ A changing magnetic field creates an electric field.

<u>Faraday's Law</u> The integral of the electric field around a closed path C is proportional to the time rate of change of the flux through the surface bounded by C

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot \hat{n}\, da$$

⟹ The curve C is fixed

⟹ The positive normal for the surface S is chosen s.t. if the fingers of the right hand curl in the direction of C, the thumb points in the direction of the positive normal.

⟹ Special case of flux rule.

If you don't want to use the RHR above to handle the signs, use Lenz' law.

Lenz' Law - The induced current (~~in the same~~ direction as induced electric field) flows in a direction through S s.t. the flux of the induced current opposes the change in flux.

Differential Form

$$\oint_C \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{a}$$

$$= -\frac{d}{dt} \int \vec{B} \cdot d\vec{a}$$

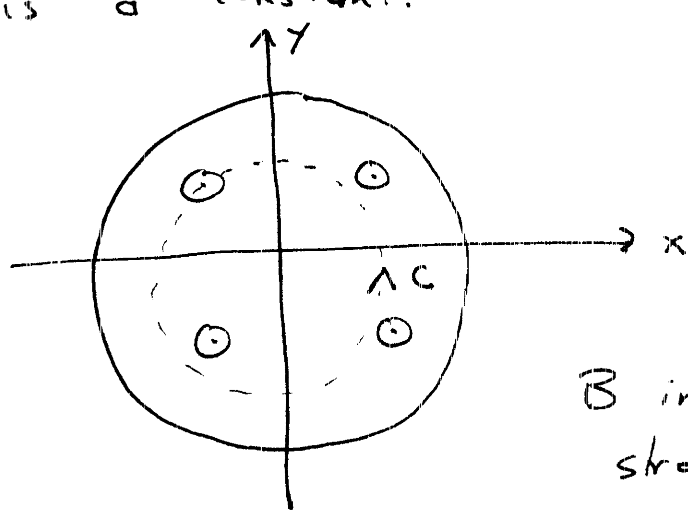
$$\Rightarrow \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$\Rightarrow$ Correct everywhere without further assumptions

$\Rightarrow$ Do not need a loop of wire.

Ex Consider a cylindrical region $s < a$ containing the magnetic field $\vec{B} = B_0 \frac{t}{\tau} \hat{z}$

where τ is a constant.



B increasing in strength with time

By RHR, positive normal is in the $+\hat{z}$ direction.

The flux through a loop of radius s, $s<a$ is

$$\Phi_m = \oint \vec{B}\cdot d\vec{a} = \pi s^2 B$$

$$= \pi s^2 B_0 \frac{t}{\tau}$$

Note, sign is correct because $\vec{B} \parallel \hat{n}$.

<u>Faradays Law</u>

$$\oint_c \vec{E}\cdot d\vec{l} = -\frac{d\Phi_m}{dt} = -\frac{\pi s^2 B_0}{\tau}$$

By symmetry, the electric field must be circles about z axis, $\vec{E} = E(s)\hat{\phi}$

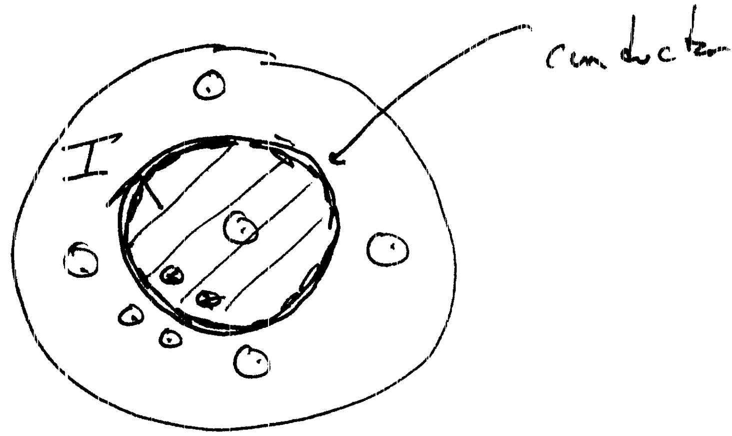
Since $d\vec{l} \parallel \vec{E}$, $\oint_c \vec{E}\cdot d\vec{l} = E\oint dl = 2\pi s E$

$$= -\frac{d\Phi_m}{dt} = \frac{-\pi s^2 B_0}{\tau}$$

$$2\pi s E = -\frac{\pi s^2 B_0}{T}$$

$$\vec{E} = \frac{-s B_0}{2T}\hat{\phi}$$

Let's check the direction by using Lenz' Law, the electric field must be in the direction of the current induced in a conducting loop placed in the field.



The flux out of the page is increasing, the induced current must produce a flux into the page inside the loop $\Rightarrow$ current is CW $\Rightarrow$ field is CW $\Rightarrow$ Field is in the negative $\hat{\phi}$ direction. ✓