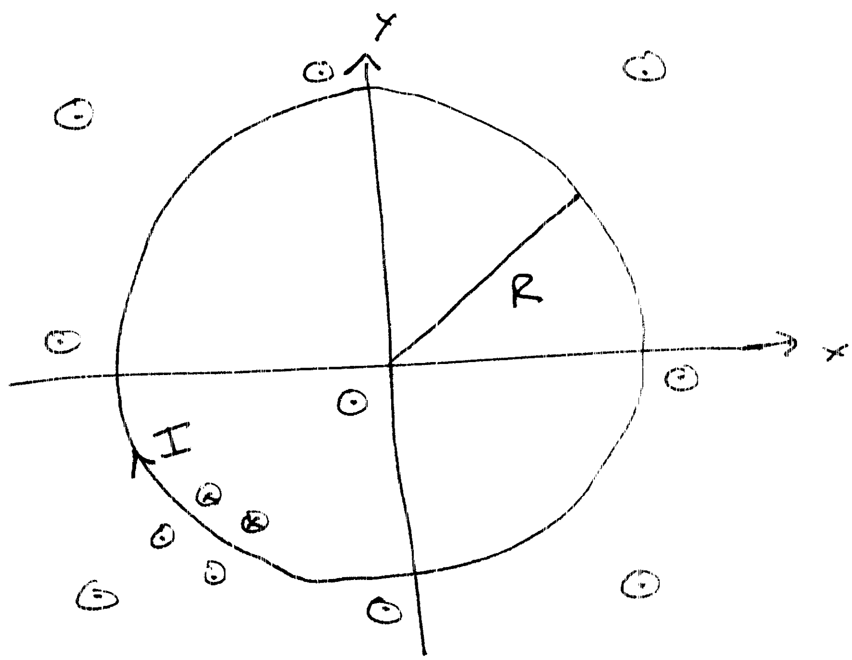
Electromotive Force II

Ex A loop of radius R and cross-sectional area A is made of a material with resistivity ρ. Compute the back field (field of induced current) at the center of the loop if it is placed in a magnetic field

$\vec{B} = \gamma t^2 \hat{z}$, γ constant.



$N = 1$

Magnetic Flux

$$\Phi_m = \int \vec{B} \cdot d\vec{a} = \vec{B} \cdot \hat{n}\, A_g = B \pi R^2$$

A_g = Area of Loop = πR^2

$$\Phi_m = \gamma \pi R^2 t$$

Flux Rule (Faraday's Law) -

$$\text{emf} = -\frac{d\Phi_m}{dt} = -2\gamma' \pi R^2 t$$

Direction of Induced Current

Method I - Assume $\vec{B}$ points in direction of positive normal. Point thumb in direction of positive normal, fingers curl in direction of positive direction of current path C, CCW. Since emf < 0, current flows clockwise.

Method II - Flux out of the page is increasing. To oppose the increasing flux, the induced current produces a flux into the page, and by the RHR a clockwise current. (Lenz Law)

The resistance of the loop is

radius

$$R_L = \frac{\rho l}{A} = \frac{2\pi R \rho}{A}$$

cross-section

The current is then,

$$I = \frac{emf}{R_L} = \frac{-2\gamma\pi R^2 t}{2\pi R\rho/A}$$

$$= -\frac{\gamma A R}{\rho} \qquad (- \text{ indicate CW direction})$$

This current produces a field into the page at the center of the loop.

$$\vec{B}_{back} = \frac{-\mu_0 I}{2R}\hat{z} = \frac{-\mu_0\gamma A R}{2R\rho}\hat{z}$$

$$= -\frac{\mu_0\gamma A}{2\rho}\hat{z}$$

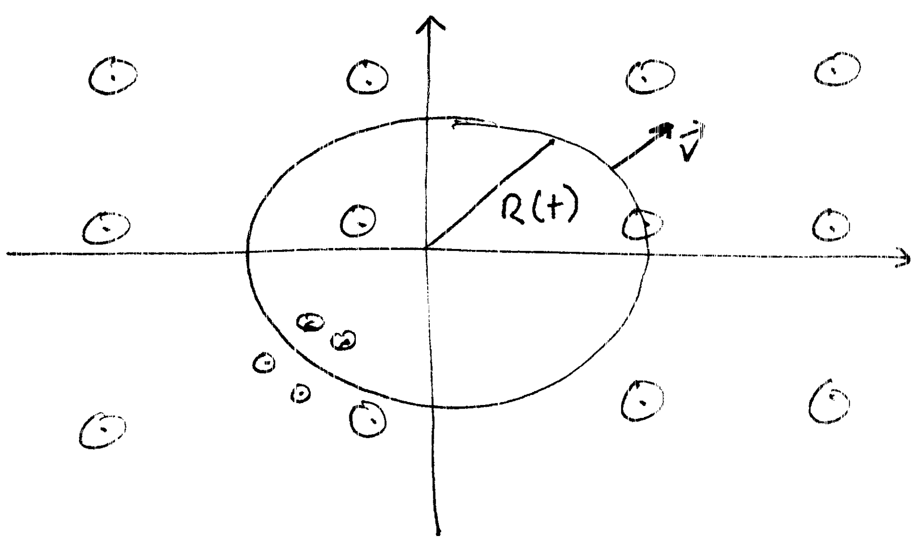
- Weirdly, back field does not depend on Radius of Loop
- Note, direction is correct because field direction taken from RHR on current.
- Note, obviously motional emf could not be used to work this problem.

Ex What would change if we tipped the loop by $30°$?

$$\Phi_m = N\vec{B}\cdot\hat{n}\,A_\ell = BA_\ell\cos 30°$$

Ex Now, put a loop in a fixed field $\vec{B} = B_0\hat{z}$, but let the radius of the loop grow as

$$R(t) = R_0 + vt$$



Magnetic Flux

$$\Phi_m = N\vec{B}\cdot\hat{n}\,A = B_0\,\pi\,R(t)^2$$

$$= B_0\pi\,(R_0+vt)^2$$

Flux Rule (Not Faraday)

$$emf = -\frac{d\Phi_m}{dt} = -2B_0\pi v\,(R_0+vt)$$

Lenz Law - Increasing flux out of the page, so induced current must produce flux into the page to oppose the change, by RHR current CW.

Compute directly from motional emf

$$\text{emf} = \frac{W}{q} = \int_C (\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$\vec{v}\times\vec{B} = |v| B_0 \hat{\phi}$$

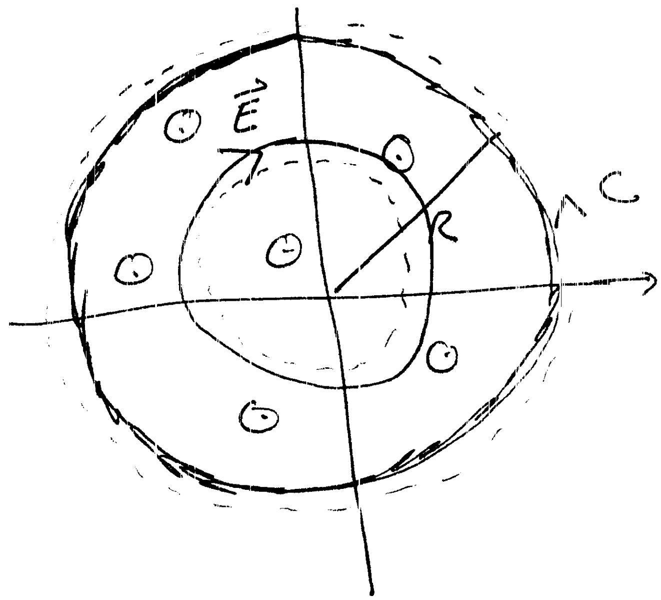
$$= -\frac{dR}{dt} B_0 \hat{\phi} = -v B_0 \hat{\phi}$$

$-\hat{\phi} = \text{CW}$

$$d\vec{l} = R\hat{\phi}\, d\phi$$

$$\text{emf} = -v B_0 R \int_0^{2\pi} d\phi = -2\pi v B_0 (R_0 + vt)$$

Returning to the first example, suppose the time varying magnetic field $B = \gamma t^2 \hat{z}$ is confined to cylindrical region. By symmetry, the electric field is circular. The field must be in the direction of the induced current.



The emf around curve C is

$$\text{emf} = \oint_C \vec{E} \cdot d\vec{l} = 2\pi R E$$

$$= -\frac{d\Phi_m}{dt} = -2\gamma\pi R^2 t$$

therefore the electric field at R is

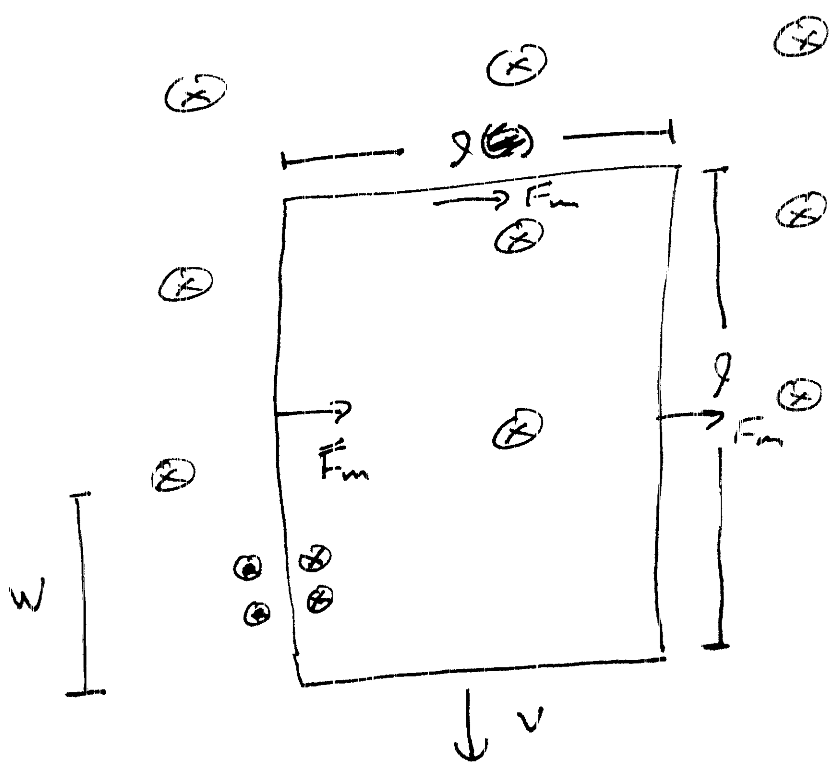
$$E = \frac{-\frac{d\Phi_m}{dt}}{2\pi R} = -\gamma R t$$

It is then trivial to show the emf

$$\text{emf} = \oint \vec{E} \cdot d\vec{l} = -2\pi \gamma R^2 t$$

is the same calculated by the flux rule.

<u>Ex</u> Loop pulled out of magnetic field $\vec{B} = B_0 \hat{z}$



Pretend square loop sides l.

$$w(t) = w_0 + vt$$

$$\Phi_m = NBl(l - w(t)) = NBl(l - w_0 - vt)$$

$$\text{emf} = -\frac{d\Phi_m}{dt} = NBvl \qquad \text{(Flux Rule)}$$

Also calculate with motional emf, only top contributes emf around loop.

$$\text{emf} = N \oint \vec{v} \times \vec{B} \cdot d\vec{l}$$

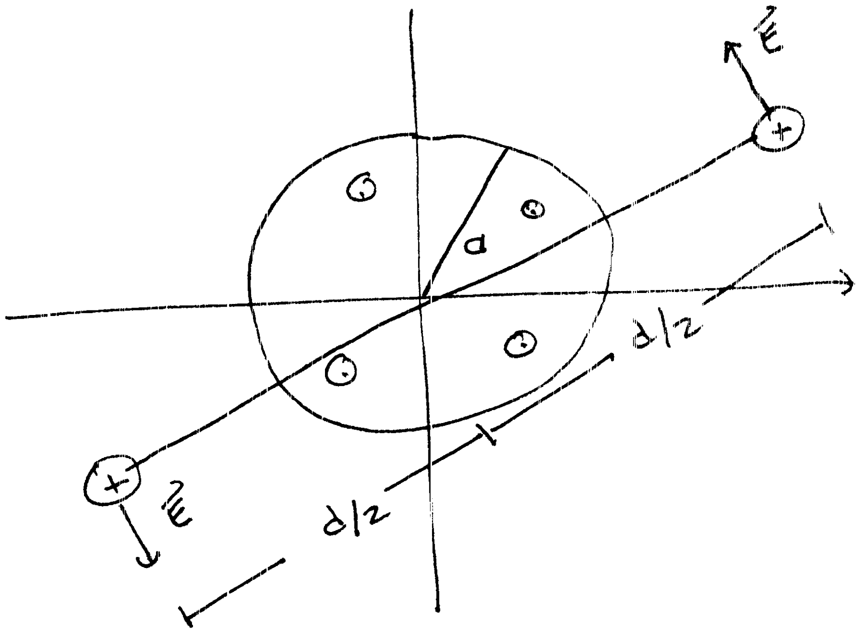
$$= N v B l$$

Motional emf clearly gives current CW.

<u>Lenz Law</u> - Flux into page is decreasing, induced current will produce field in same direction to oppose change, by RHR current CW.

⟹ Note, if $\vec{B} = B_0 t \hat{z}$ motional emf and flux rule give different results, flux rule is correct.

Ex <u>Crazy GRE Question</u> Let region $s < a$ contain magnetic field $\vec{B} = B_0 \hat{z}$. The field decays from B_0 to 0 in time Δt. Compute the change in angular momentum of two charges on a stick of length $d > 2a$.



Torque τ -

$$\tau = \frac{dL}{dt}$$

where L is angular momentum.

$$\Delta L = \int_0^{\Delta t} \tau \, dt$$

By symmetry, field is circular. The torque is then

$$\tau = 2\tau_1 = 2\left(\underbrace{\frac{d}{2}}_{\text{moment arm}} \underbrace{q E(d/2)}_{\text{force}}\right)$$

$$\tau = d q E(d/2)$$

<u>Find $\vec{E}$</u>

$$\text{emf} = \oint \vec{E} \cdot d\vec{l} = 2\pi s E(s) = -\frac{d\Phi_m}{dt}$$

$$\Phi_m = NBA = \quad ^2 = B_0 \pi a^2$$

$$\frac{d\Phi_m}{dt} = \pi a^2 \frac{dB}{dt}$$

$$2\pi s E(s) = -\pi a^2 \frac{dB}{dt}$$

$$E(s) = -\frac{a^2}{2s} \frac{dB}{dt}$$

At $d/2$

$$E(d/2) = \frac{-a^2}{d} \frac{dB}{dT}$$

and the torque is

$$\tau = d\,q\,E(d/2) = -q a^2 \frac{dB}{dT}$$

The change in angular momentum is

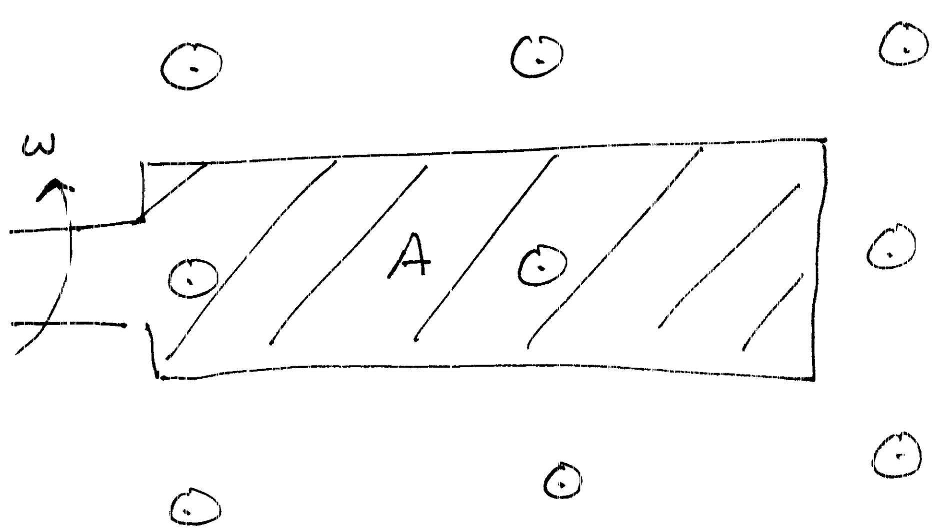
$$\Delta L = \int \tau \, dt = -q a^2 \int dB$$

$$= -q a^2 \Delta B = -q a^2 B_0$$

Electromotive Force III

<u>Generators</u> - Convert mechanical energy to electrical energy

$\Rightarrow$ Normally loop spinning in magnetic field



$$\Phi_m = N \int \vec{B} \cdot \hat{n}\, da = N \vec{B} \cdot \hat{n} A$$

$$= NBA \cos \theta(t)$$

Period (T) - Time for one rotation

Frequency (f) - Rotations per second

Angular Frequency (ω) - $2\pi f = \omega$

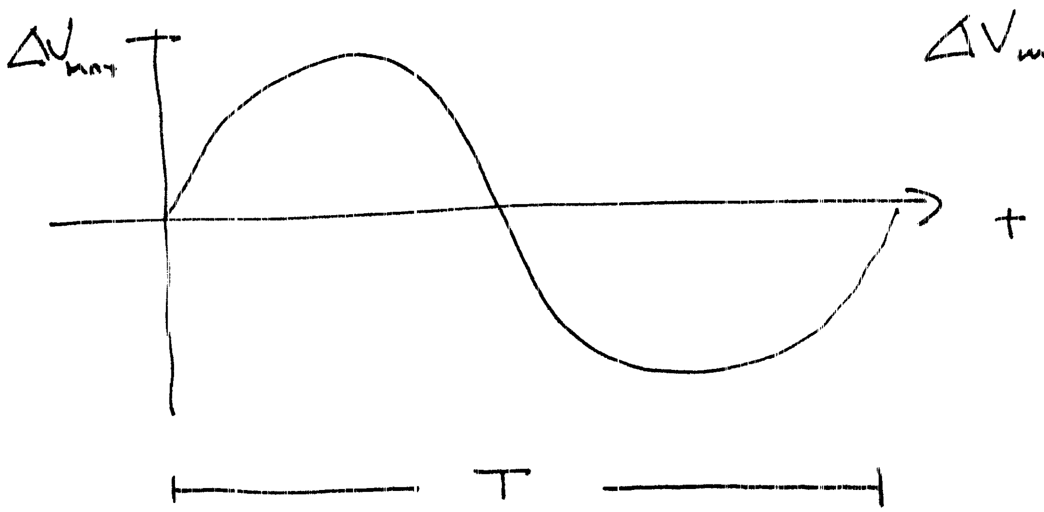
If ω constant, $\theta(t) = \theta_0 + \omega t$

$$\Phi_m = NBA\cos(\omega t + \theta_0)$$

<u>Faraday's Law</u>

$$emf = -\frac{d\Phi_m}{dt} = NBA\omega\sin(\omega t + \theta_0)$$

$$= \Delta V_{max}\sin(\omega t + \theta_0)$$

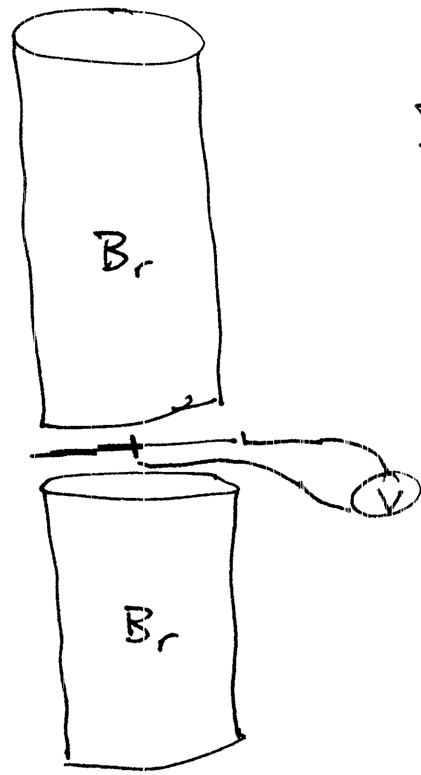


$\Delta V_{max} = NBA\omega$

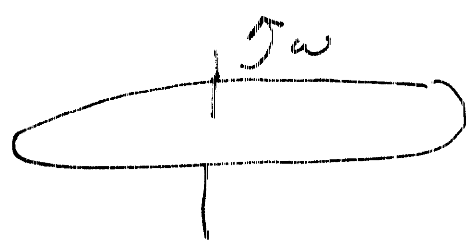
if $\theta_0 = 0$

Many other designs are possible

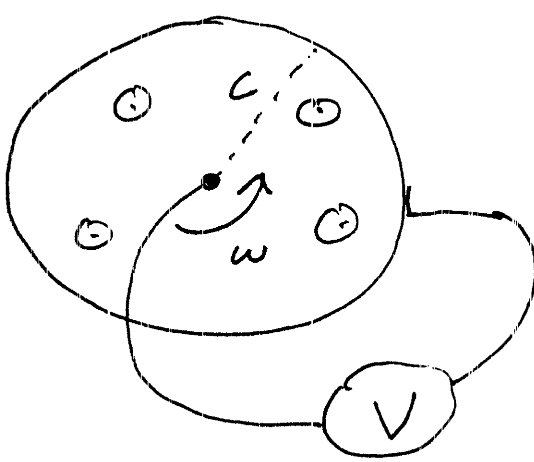
Faraday's Dynamo



Disk spinning in permanent magnet.



Top View



Radius R

Use motional emf along dashed line

$$\text{emf} = \int_c (\vec{v} \times \vec{B}) \cdot d\vec{l} \qquad d\vec{l} = ds\,\hat{s}$$

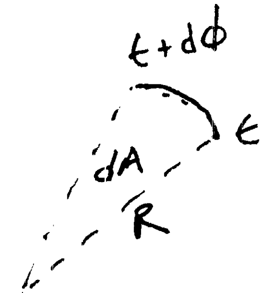
$$\vec{v} = \omega s\,\hat{\phi} \qquad \vec{v} \times \vec{B} = \omega s B_r\,\hat{s}$$

$$\text{emf} = \int_c (\vec{v} \times \vec{B}) \cdot d\vec{l} = \omega B_r \int_0^R s\,ds$$

$$= \frac{\omega B_r R^2}{2}$$

<u>Check with flux rule</u> - In time dt

a wedge



of area $dA = \frac{1}{2} R \cdot R\,d\phi$ is swept out.

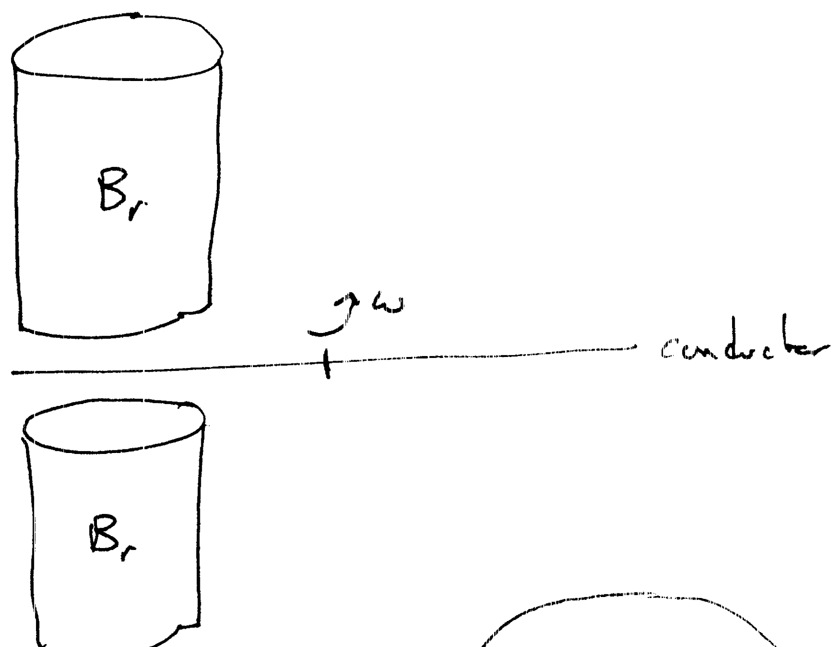
$$d\Phi_m = W B_r\,dA = \frac{B_r R^2}{2}\,d\phi$$

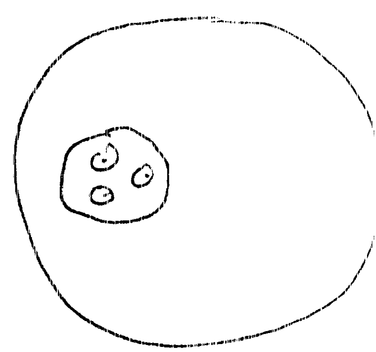
$$|\text{emf}| = \left| -\frac{d\Phi_m}{dt} \right| = \frac{B_r R^2}{2}\frac{d\phi}{dt} = \frac{B_r R^2 \omega}{2}$$

The energy lost to heat comes from the kinetic energy of the disk, and therefore the disk slows $\Longrightarrow$ Magnetic Braking.

If no load is connected to the dynamo, a region of positive charge will result on the outer edge after a brief time, current flow will stop, and the disk will spin without additional input of energy.

<u>Eddy Currents</u> - Consider a different system





Now the flux through a portion of the conductor is constantly changing, producing eddy current. These currents cause Joule heating.