Electricity and Magnetism - Final Exam - Spring 2010

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

Problem 4.1 Show how the divergence of the Poynting vector,

$$\nabla \cdot \vec{S} = \nabla \cdot \frac{\vec{E} \times \vec{B}}{\mu_0}$$

is related to the electric and magnetic energy densities $u_e = \frac{1}{2}\epsilon_0 \vec{E} \cdot \vec{E}$ and $u_m = \frac{1}{2}\vec{B} \cdot \vec{B}/\mu_0$. Cite any law or vector identity used. Note, you do not need to know anything about the Poynting vector to do this problem. Assume $\vec{J} = 0$.

Problem 4.2 The $x - y$ plane is the boundary between two regions $z > 0$ and $z < 0$ with different electric and magnetic fields. The fields below the plane are

$$\vec{E}_- = \gamma(\hat{x} + 2\hat{y})$$

and

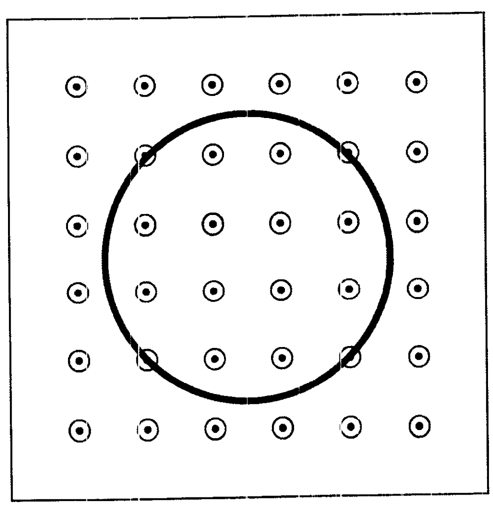
$$\vec{B}_- = \alpha(\hat{x} + \hat{z})$$

The plane has a surface charge density σ and a surface current density $\vec{K} = \Gamma\hat{y}$. Γ, γ, and α are constants. Find the electric and magnetic field above the place ($z > 0$).

Problem 4.3 A solenoid of radius b has N turns wound over ℓ distance and carries current I. The solenoid contains a hollow cylindrical iron core with relative permeability μ_r at the operating field. The iron core has inner radius a and outer radius b. Compute $\vec{B}$, $\vec{H}$, $\vec{M}$, $\vec{J}_b$, and $\vec{K}_b$ everywhere.

Problem 4.4 A spherical system of radius a has potential $V(a, \theta) = \gamma \cos(\theta)$ at its surface. Compute the electric field inside the system.

Problem 4.5 A circular ring of conducting wire is in a region with changing magnetic field as shown below. The radius of the ring is increasing as $a(t) = a_1 t^2$, where a_1 is constant. The magnetic field is $B(t) = B_0 t^2$ in the direction drawn with B_0 constant. The conductivity of the wire is σ and it has cross-sectional area A_w. Compute the current flowing the the wire as a function of time and give its direction.



Problem 4.6 A spherical system is built with a free charge density $\rho(r) = \gamma r$ ($r < a$) embedded in a linear dielectric material of radius a with relative permittivity ϵ_r. Calculate the potential difference between the center of the system and the surface ΔV_{0a}. γ is constant.

1

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Griffith's Product Rule (6)

$$\nabla \cdot \vec{S} = \frac{1}{\mu_0} \nabla \cdot (\vec{E} \times \vec{B})$$

$$= \frac{1}{\mu_0} \left[\vec{B} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{B} \right]$$

$$= \frac{1}{\mu_0} \left[\vec{B} \cdot \left(-\frac{\partial \vec{B}}{\partial t} \right) - \vec{E} \cdot \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \right]$$

using Faraday

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and Ampere ($\vec{J} = 0$)

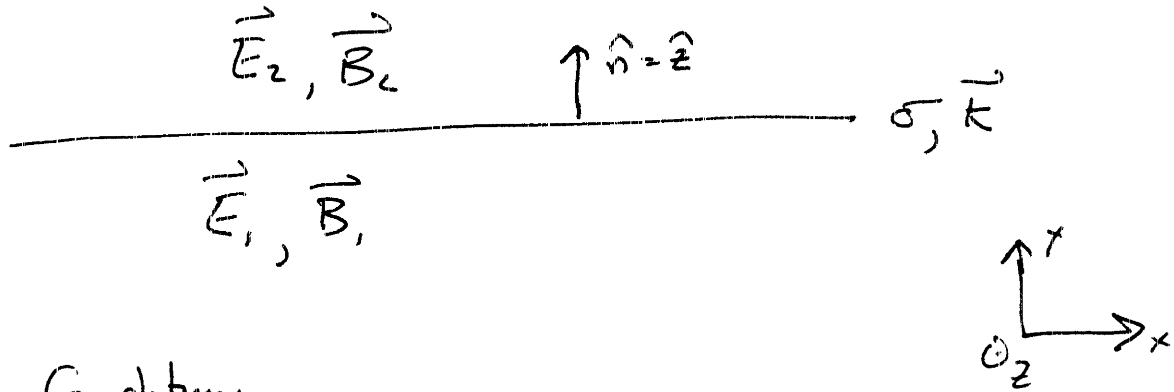
$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\frac{\partial \vec{B} \cdot \vec{B}}{\partial t} = 2 \vec{B} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{S} = \frac{1}{\mu_0} \left[-\frac{1}{2} \frac{\partial (\vec{B} \cdot \vec{B})}{\partial t} - \frac{\mu_0 \epsilon_0}{2} \frac{\partial \vec{E} \cdot \vec{E}}{\partial t} \right]$$

$$= -\frac{\partial u_m}{\partial t} - \frac{\partial u_e}{\partial t}$$

2



<u>Boundary Conditions</u>

$$E_2^{\perp} - E_1^{\perp} = \sigma/\epsilon_0$$

$$E_{2z} = E_{1z} + \sigma/\epsilon_0 = \sigma/\epsilon_0$$

$$\vec{E}_2^{\parallel} = \vec{E}_1^{\parallel} \qquad \vec{E}_2^{\parallel} = \gamma\hat{x} + 2\gamma\hat{y}$$

$$\boxed{\vec{E}_2 = \gamma\hat{x} + 2\gamma\hat{y} + \frac{\sigma}{\epsilon_0}\hat{z}}$$

$$B_2^{\perp} = B_1^{\perp} \Rightarrow B_2^{\perp} = \alpha\hat{z}$$

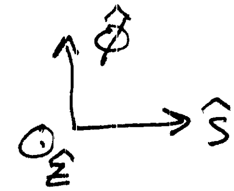
$$\vec{B}_2^{\parallel} - \vec{B}_1^{\parallel} = \mu_0(\vec{K} \times \hat{n}) = \mu_0(\Gamma\hat{y} \times \hat{z}) = \mu_0\Gamma\hat{x}$$

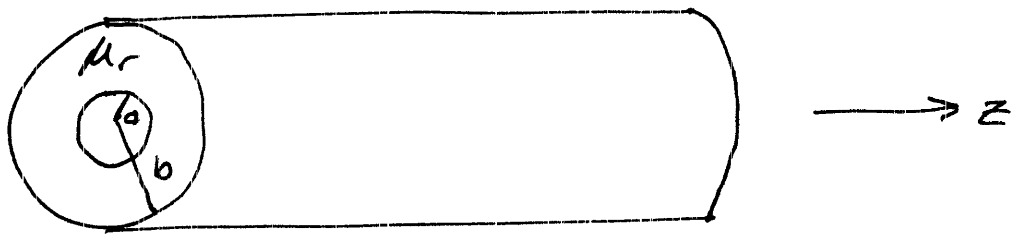
$$\vec{B}_2^{\parallel} = \vec{B}_1^{\parallel} + \mu_0\Gamma\hat{x} = \alpha\hat{x} + \mu_0\Gamma\hat{x} = (\alpha + \mu_0\Gamma)\hat{x}$$

$$\vec{B}_2 = (\alpha + \mu_0\Gamma)\hat{x} + \alpha\hat{z}$$

4.3

$n = \frac{N}{\ell}$





The magnetic field of an infinite solenoid is

$$\vec{B}_i = \mu_0 n I \hat{z} \quad , \quad \vec{B}_0 = 0$$

ignoring the iron. Therefore, the $\vec{H}$ field is

$s < b$, $\quad \vec{H} = \frac{\vec{B}_i}{\mu_0} = n I \hat{z}$

$s > b \quad \vec{H} = 0 \Rightarrow \vec{B} = 0 \Rightarrow \vec{M} = 0$

The introduction of the iron does not change the field.

For $s < a$, $\quad \vec{M} = 0$

$$\vec{B} = \mu_0 \vec{H} = \mu_0 n I \hat{z}$$

For $a < s < b$,

$$\vec{B} = \mu_0 \mu_r \vec{H} = \mu_0 \mu_r n I \hat{z}$$

The magnetization is $\vec{M} = \chi_m \vec{H} = (\mu_r - 1)\vec{H}$

$$= (\mu_r - 1)\, n I\, \hat{z}$$

The bound current density

$$\vec{J}_b = \nabla \times \vec{M} = 0$$

Surface current density, $s = a$ surface

$$\hat{n} = -\hat{s}$$

$$\vec{K}_b = \vec{M} \times (-\hat{s}) = -(\mu_r - 1)\, n I\, (\hat{z} \times \hat{s})$$

$$= -(\mu_r - 1)\, n I\, \hat{\phi}$$

Surface current density, $s = b$, $\hat{n} = +\hat{s}$

$$\vec{K}_b = \vec{M} \times \hat{s} = (\mu_r - 1)\, n I\, \hat{\phi}$$

Note, surface currents enhance field in iron

4.4 Since the system is spherical, the general solution of Laplace's eqn is

$$V(r,\theta) = \sum_n \left(A_n r^n + B_n r^{-(n+1)} \right) P_n(\cos\theta)$$

For the solution to be finite, $B_n = 0$.

The potential at the surface is given as

$$V(a,\theta) = \Gamma \cos\theta = \Gamma P_1(\cos\theta)$$

By orthogonality, $A_n = 0$ if $n \neq 1$.

$$V(a,\theta) = A_1 a^1 P_1(\cos\theta) = \Gamma P_1(\cos\theta)$$

$$A_1 = \frac{\Gamma}{a}$$

The potential is then

$$V(r,\theta) = \frac{\Gamma r}{a} P_1(\cos\theta) = \frac{\Gamma r}{a} \cos\theta$$

The electric field inside the sphere is

$$\vec{E} = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}\right]$$

$$\vec{E} = -\left[\frac{\Gamma}{a}\cos\theta\,\hat{r} + \frac{1}{r}\cdot\frac{\Gamma r}{a}\cdot -\sin\theta\,\hat{\theta} + 0\right]$$

$$= -\frac{\Gamma}{a}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right] = -\frac{\Gamma}{a}\hat{z}$$

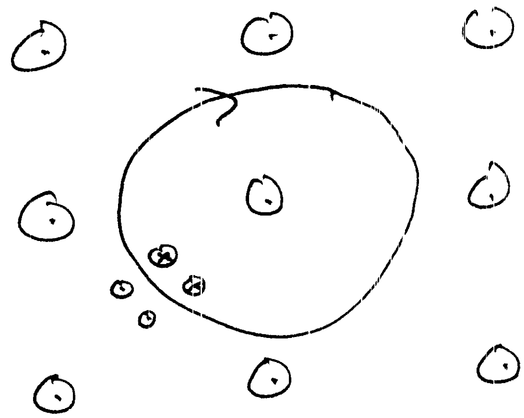
We could get this more immediately by recognizing

$$z = r\cos\theta$$

$$V = \frac{\Gamma}{a} z$$

$$\vec{E} = -\nabla V = -\frac{\Gamma}{a}\hat{z}$$

(4.5) To oppose an increasing flux out of the page, the induced current must flow clockwise



The resistance of the loop is $R = \frac{2\pi a}{\sigma A_w}$

The flux through the loop is

$$\Phi_m = BA = (B_0 t^2)\,\pi\,(a_1 t^2)^2$$

$$= B_0 a_1^2 \pi t^6$$

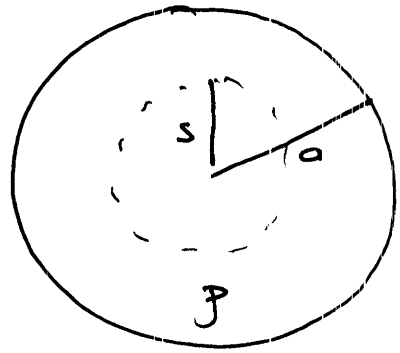
The emf around the loop is

$$emf = -\frac{d\Phi_m}{dt} = -6 B_0 a_1^2 \pi t^5$$

The current is then

$$I = \frac{emf}{R} = \frac{6 B_0 a_1^2 \pi t^5}{2\pi a_1 t^2/\sigma A_w} = 3\sigma A_w a_1 t^3 B_0$$

4.6



The charge enclosed in a spherical Gaussian surface of radius r

$$Q_{fenc} = \int_0^r 4\pi r^2 \rho \, dr = \int_0^r 4\pi r^2 \gamma r \, dr$$

$$= 4\pi\gamma \frac{r^4}{4} = \pi\gamma r^4$$

The displacement field is then

$$\oint \vec{D} \cdot d\vec{a} = 4\pi r^2 D = Q_{fenc}$$

$$\vec{D} = \frac{\pi\gamma r^4}{4\pi r^2}\hat{r} = \frac{\gamma r^2}{4}\hat{r}$$

Since the material is linear,

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}$$

and

$$\vec{E} = \frac{\gamma r^2}{4 \varepsilon_r \varepsilon_0} \hat{r}$$

The potential difference between the center and the surface is

$$\Delta V_{0a} = -\int_{0 \to a} \vec{E} \cdot d\vec{l} = -\int_0^a \frac{\gamma r^2}{4 \varepsilon_r \varepsilon_0} \hat{r} \cdot \hat{r} \, dr$$

$$= \frac{-\gamma}{4 \varepsilon_r \varepsilon_0} \int_0^a r^2 = \frac{-\gamma}{4 \varepsilon_r \varepsilon_0} \frac{a^3}{3}$$

$$= -\frac{\gamma a^3}{12 \varepsilon_r \varepsilon_0}$$