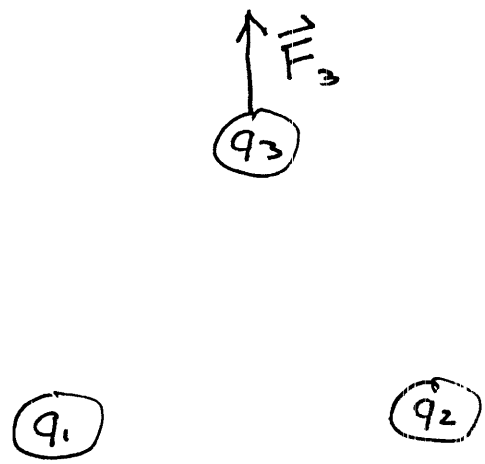
Force and Pressure

For a system of point charges, q_1, q_2, q_3, we have a number of options for calculating the force on one of the charges, say q_3.

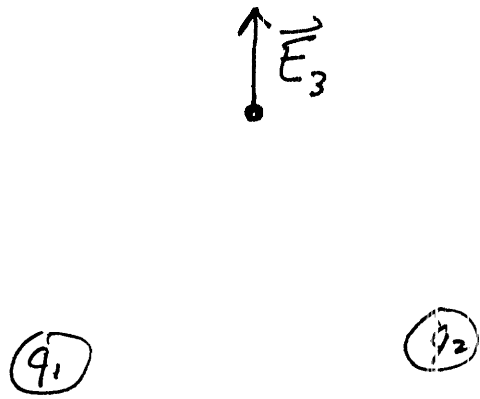


I. Calculate the force q_1 and q_2 exert on q_3 and add.

$$\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23}$$

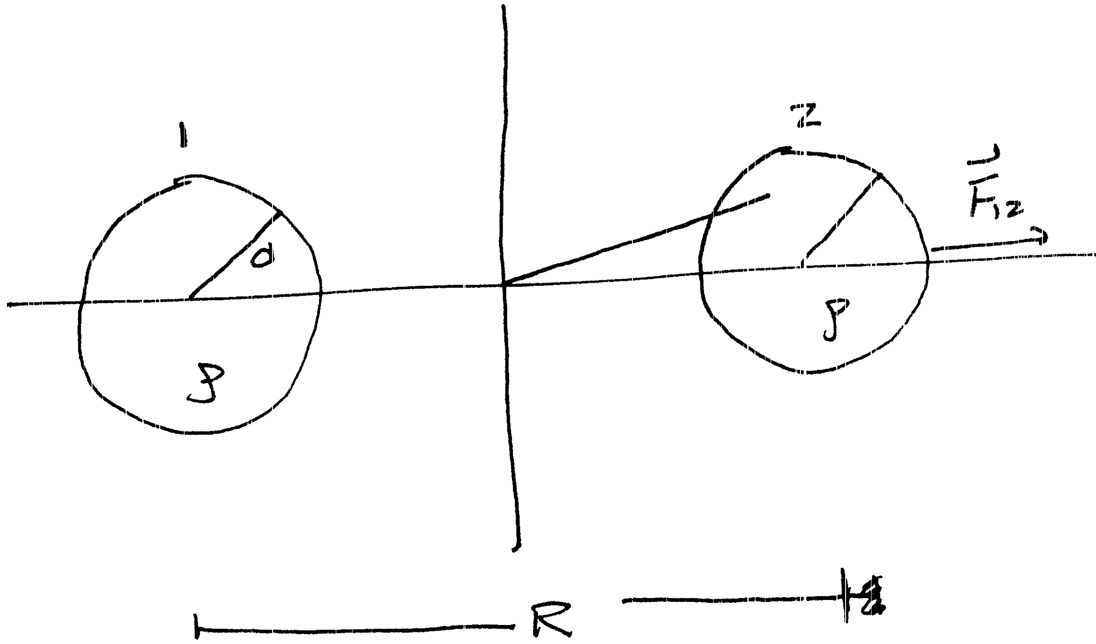
- Note, we do not include $\vec{F}_{33}$, the force q_3 exerts on itself, $\vec{F}_{33} = 0$, since all internal forces come in pairs (Newton III).

II. We can calculate the field due to q_1 and q_2 at the location of q_3, sum to get the total field, $\vec{E}_3$, then $\vec{F}_3 = q_3 \vec{E}_3$.



This calculation is done by pretending q_3 is not present.

Extended systems of charge also exert forces on one another. Consider two spheres with uniform charge density ρ a distance R apart.



How would we calculate the total force sphere 1 exerts on sphere 2?

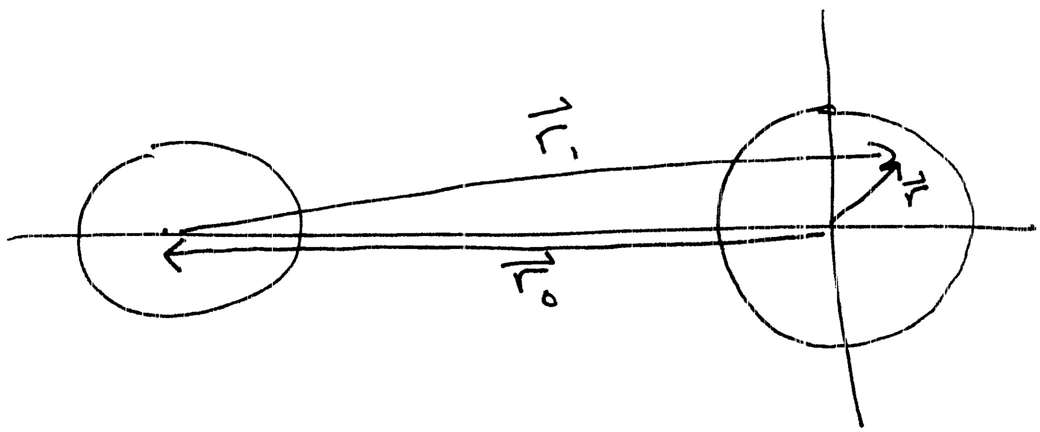
<u>Method I</u> Brute force (always works, sometimes difficult)

$$\vec{F}_{12} = \underbrace{\int_{V_1} \rho\, d\tau_1}_{q_1} \underbrace{\int_{V_2} \rho\, d\tau_2}_{q_2} \frac{\hat{r}_{12}}{4\pi\epsilon_0 r_{12}^2}$$

⟹ It always works to divide the system into point charge like things.

⟹ That integral looks pretty hard.

Method II - Calculate the field of sphere 1 at sphere 2, then integrate $\vec{F}_{12} = dq_2 \vec{E}$ to get force (4)



$$\vec{E} = \frac{Q_T}{4\pi\varepsilon_0 r_1^2}\hat{r}_1 \qquad Q_T = \frac{4}{3}\pi a^3 \rho$$

$$-\vec{r}_0 + \vec{r} = +\vec{r}_1$$

$$\vec{r}_0 = -R\hat{x}$$

$$\vec{r}_1 = +R\hat{x} + \vec{r} \qquad \vec{r}_1 = (x+R, y, z)$$

$$\vec{E} = \frac{Q_T}{4\pi\varepsilon_0 r_1^3}\vec{r}_1 = \frac{Q_T}{4\pi\varepsilon_0 \left((x+R^2) + y^2 + z^2\right)^{3/2}}(x+R, y, z)$$

It sure looks like we should do the integral in spherical. Some symmetry first, the force must point in $+\hat{x}$ direction.

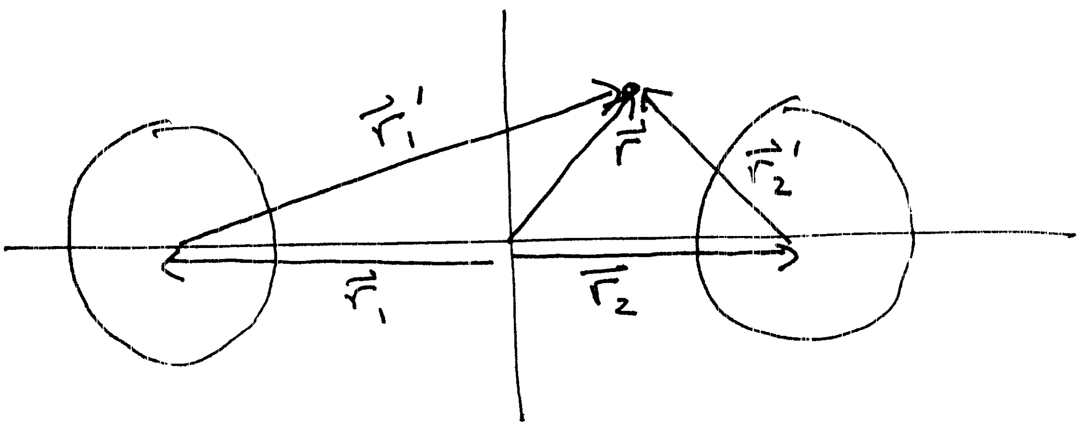
$$\vec{F}_{12} = \int d\tau_2 \, \rho \, \vec{E}_{12}(\vec{r})$$

$$= \frac{Q_T \hat{x} \rho}{4\pi\epsilon_0} \int_{V_2} \frac{d\tau_2 \, (x+R)}{((x+R)^2+y^2+z^2)^{3/2}}$$

$$= \frac{Q_T \hat{x} \rho}{4\pi\epsilon_0} \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz$$

$$\times \frac{x+R}{((x+R)^2+y^2+z^2)^{3/2}}$$

Method III – Calculate the total field of both objects and integrate $dq\,\vec{E}$.



In this case we exclude dq from the calculation of the field but since it is an infinitesimal it does not change the field.

$$\vec{r}_1' = \vec{r} - \vec{r}_1$$

$$\vec{r}_2' = \vec{r} - \vec{r}_2$$

Outside both spheres

$$\vec{E} = \frac{Q_{1T}\,(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0\,|\vec{r}-\vec{r}_1|^3} + \frac{Q_{2T}\,(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0\,|\vec{r}-\vec{r}_2|^3}$$

Inside Sphere II

$$\vec{E}_2 = \frac{\rho r_2' \hat{r}_2'}{3\varepsilon_0} = \frac{\rho \vec{r}_2'}{3\varepsilon_0}$$

$$= \frac{\rho}{3\varepsilon_0}(\vec{r} - \vec{r}_2)$$

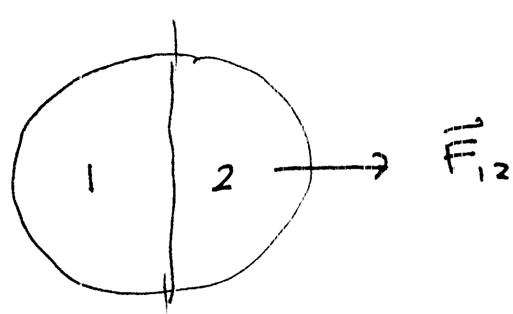
$$\vec{E} = \frac{Q_{1T}(\vec{r} - \vec{r}_1)}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_1|^3} + \frac{\rho}{3\varepsilon_0}(\vec{r} - \vec{r}_2)$$

But that has an extra term when we integrate

$$\vec{F}_{12} = \int_{V_2} \rho \, d\tau \, E$$

- The term integrates out because of Newton III.

This looks like a stupid way to do things, but consider the problem of the force one half of a sphere exerts on the other half.



The field of the half-sphere is complicated and the direct force integral undoable. But the field of the full sphere is easy from Gauss –

Ex Field of uniform volume charge inside the charge.

$$Q_{enc} = \frac{4}{3}\pi r^3 \rho$$

$$\underline{\Phi} = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} = \frac{4}{3\varepsilon_0}\pi r^3 \rho$$

$$E = \frac{r\rho}{3\varepsilon_0}$$

$$\vec{E} = \frac{r\rho}{3\varepsilon_0}\hat{r} = \frac{\rho\vec{r}}{3\varepsilon_0}$$

The force is just

$$\vec{F} = \int_{V_2} dq\,\vec{E} = \int_{V_2} \rho\, d\tau\, E$$

$$= \int_0^{\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi \int_0^{a} r^2 \sin\phi\, dr\, \rho \cdot \left(\frac{\rho\vec{r}}{3\varepsilon_0}\right)$$

Again only x component survives, $\vec{r} = (x, y, z)$

$$\vec{F} = \frac{\hat{x}\rho^2}{3\varepsilon_0} \int_0^{\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi \int_0^a r^2 \sin\theta \, x \, dr$$

θ

$$x = r\sin\theta\cos\phi \qquad (\text{Back cover})$$

$$\vec{F} = \frac{\hat{x}\rho^2}{3\varepsilon_0} \int_0^{\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi \int_0^a r^3 \sin^2\theta \cos\phi \, dr$$

The ~~quatity~~ quantity $dq\,\vec{E}$ is a force per unit volume. We can also let $dq = \sigma da$ and form a force per unit area, an electrostatic pressure.

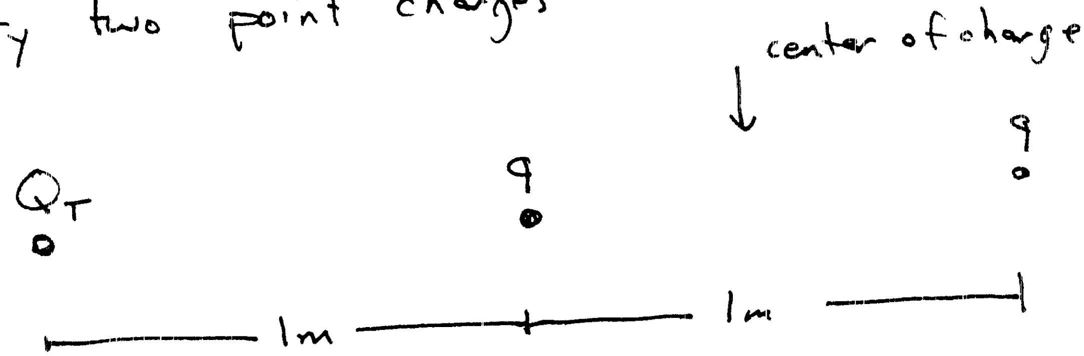
Are we working way too hard?

Is the force just $\vec{F} = \dfrac{k Q_{T_1} Q_{T_2}}{R^2} \hat{x}$

the field at the center of charge multiplied by the total charge?

This would be correct if the field was gravity near the earth or any uniform field.

Try two point charges



If we can concentrate all charge at the center of charge $r_c = 3/2\,m$ the force would be

$$F = \frac{k Q_T (2q)}{(3/2m)^2} = k Q_T q \cdot \frac{8}{9}$$

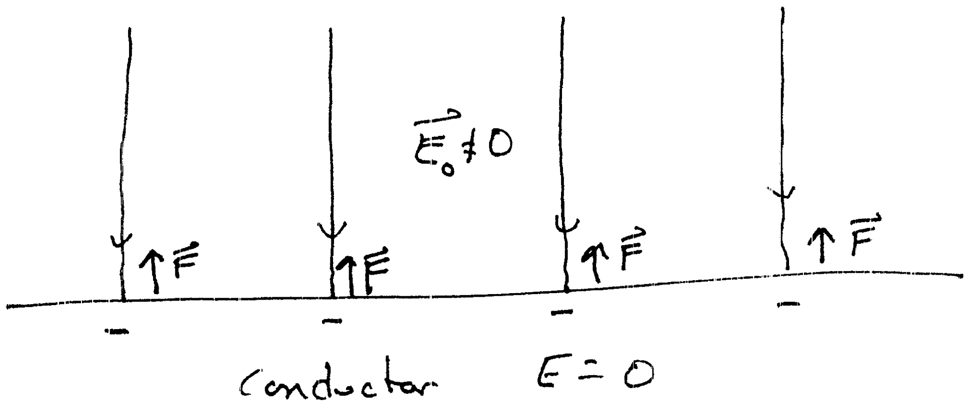
The actual force is

$$F = \frac{kqQ_T}{(1m)^2} + \frac{kqQ_T}{(2m)^2}$$

$$= kqQ_T \cdot 5/4$$

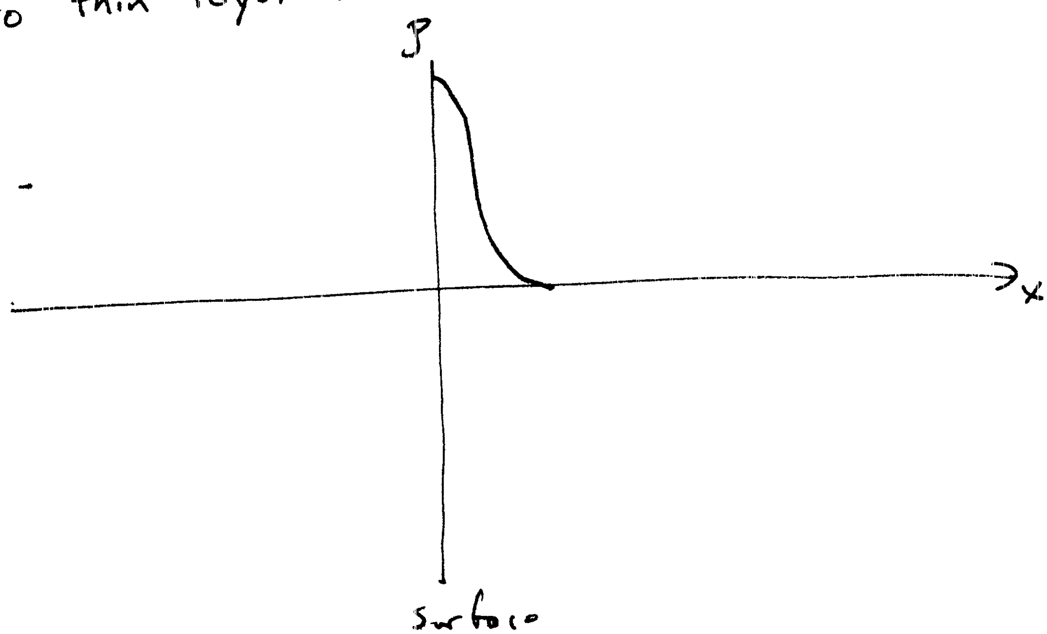
So if field is non-uniform we must integrate the force.

Consider a conductor in a uniform electric field



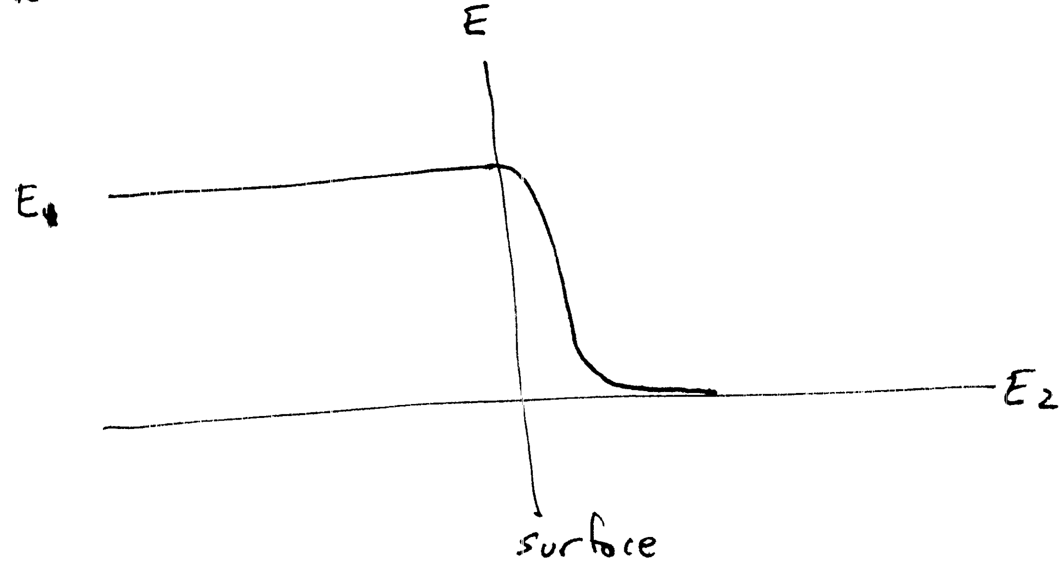
The field exerts a force on the surface charge, but what is the field at the surface? On one side the field is zero on the other $\vec{E}_0$.

Consider the real case where the charge is localized to thin layer near the surface



$$\sigma = \int \rho \, dx$$

The field also rapidly decays to zero, $\vec{E} = E(x)$



The actual force per unit area is

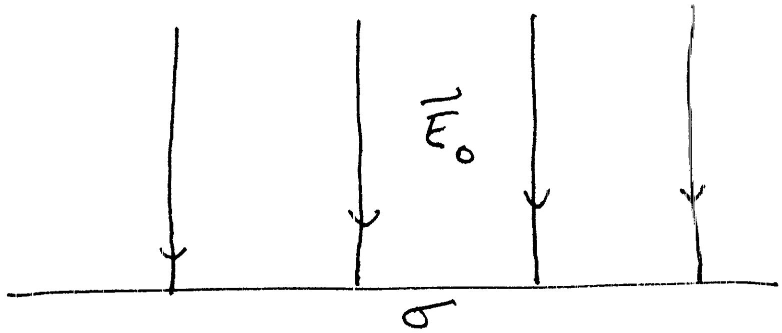
$$F = \int \rho E \, dx \approx \left(\int \rho \, dx \right) \frac{E_1 + E_2}{2}$$

So the force per unit area exerted on a surface charge density is

$$P = \sigma E_{ave} \quad - \text{ electrostatic pressure}$$

where E_{ave} is the average of the fields on the two sides of the surface.

For our original problem,



$$\text{Pressure} = P = \frac{\sigma E_0}{2}$$