Gauss' Law - Integral Form

The differential form of Gauss' Law

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

and its related equation in terms of potential

$$\nabla^2 V = -\rho/\epsilon_0$$

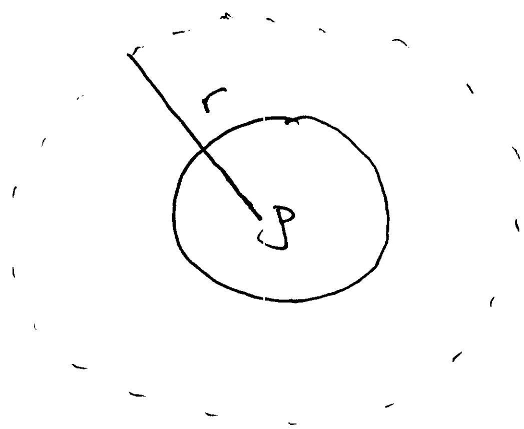
are usually the place to start when calculating the field. The integral form is completely equivalent and in some very narrow cases more powerful.

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho \, d\tau$$

<u>Gaussian Surface</u> - A surface s.t. the field is either zero or // to $\hat{n}$ and constant.

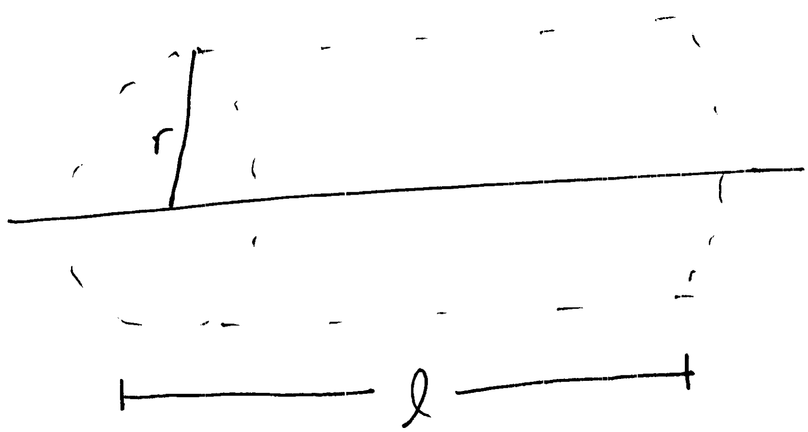
Spherically Symmetric Systems

Use spherical Gaussian surface

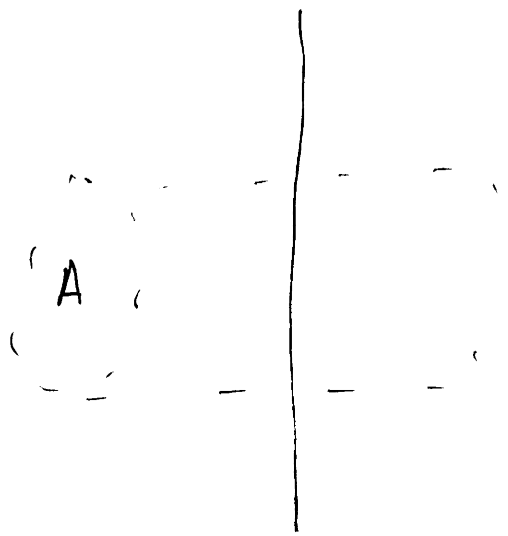


Cylindrically Symmetric Systems

Use cylindrical Gaussian surface of radius r and length ℓ.



Planar Systems (Translationally Symmetric) - Also use cylindrical surface with end area A.



For a symmetric system where a Gaussian surface exists

$$\oint_S \vec{E} \cdot d\vec{a} = \int_{S \neq 0} \vec{E} \cdot d\vec{a} + \int_{S=0} 0 \cdot d\vec{a}$$

$$= E \int_{S \neq 0} da = E A_{S \neq 0} = \Phi \text{ flux}$$

where $A_{S \neq 0}$ = area of the surface where the field is non-zero.

$$\oint_S \vec{E} \cdot d\vec{a} \equiv \Phi = E A_{s \neq 0} = \frac{Q_{enc}}{\varepsilon_0}$$

flux through S

Spherical Systems

$$\Phi = \underbrace{4\pi r^2}_{A_{s \neq 0}} E = \frac{Q_{enc}}{\varepsilon_0}$$

Cylindrical Systems

Φ out ends

$$\Phi = 2\pi r l E = \frac{Q_{enc}}{\varepsilon_0}$$

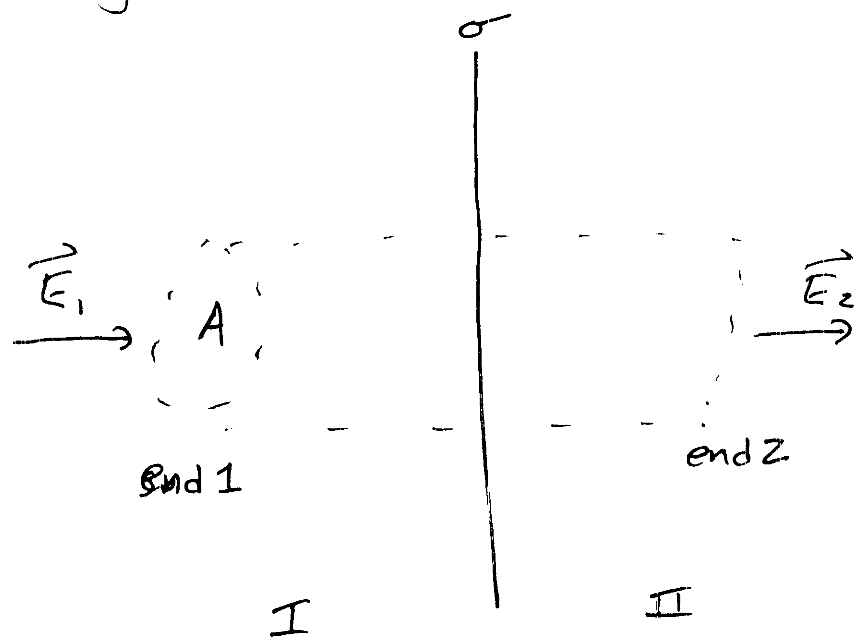
Planar Systems

$\Phi = 0$ out sides

$$\Phi = E_{out, left} A + E_{out, right} A = \frac{Q_{enc}}{\varepsilon_0}$$

careful of signs

Planar System A plane with uniform surface charge density σ is in the y-z plane



Method I Define $\vec{E}_I = E_I \hat{x}$, $\vec{E}_{II} = E_{II} \hat{x}$

$\vec{E}_I$ points to right if E_I positive, to left if E_I negative.

$$\Phi = E_{out,1} A + E_{out,2} A \qquad (\text{sides zero})$$

$$= -E_1 A + E_2 A$$

if $E_1 > 0$, field points into surface $\Rightarrow \Phi < 0$.

Gauss' Law $\quad Q_{enc} = \sigma A$

$$-E_1 A + E_2 A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

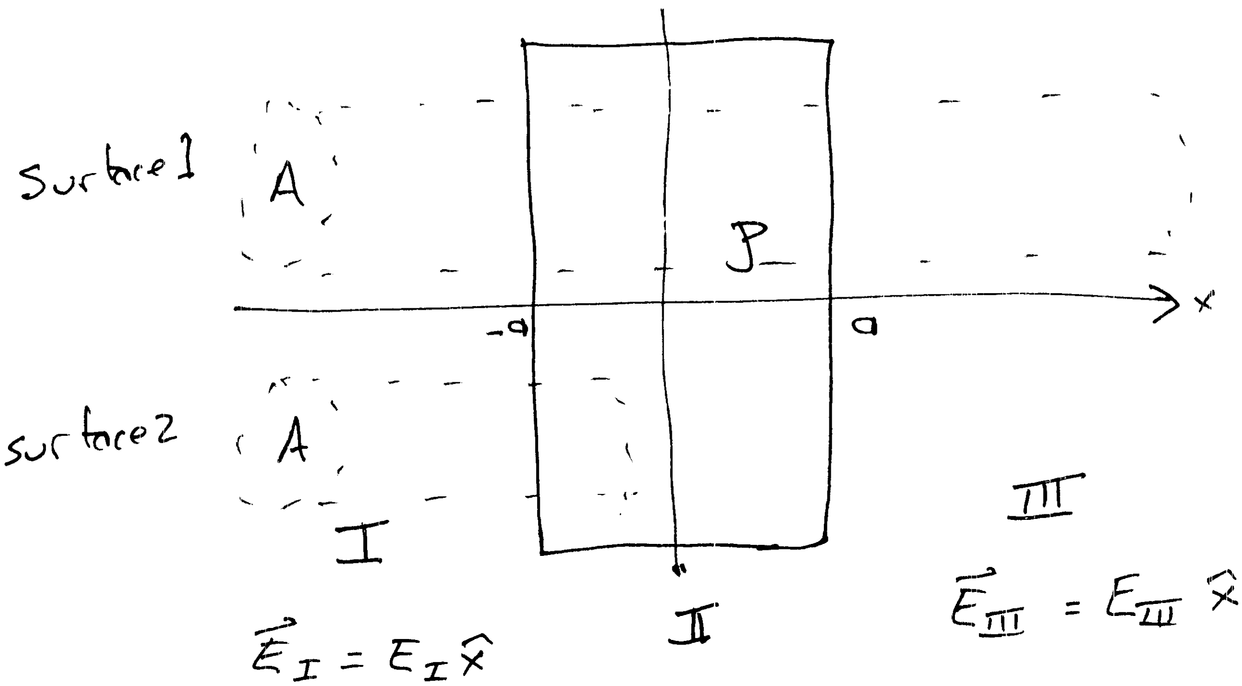
Symmetry $\quad E_2 = -E_1 \quad$ outermost fields equal but opposite.

$$-E_1 A + (-E_1)A = \frac{\sigma A}{\varepsilon_0}$$

$$E_1 = -\frac{\sigma}{2\varepsilon_0} \qquad E_2 = \frac{\sigma}{2\varepsilon_0}$$

$$\vec{E}_1 = -\frac{\sigma}{2\varepsilon_0}\hat{x} \qquad \vec{E}_2 = \frac{\sigma}{2\varepsilon_0}\hat{x}$$

Ex Try something harder — A planar slab of charge with volume charge density ρ in region $-a < x < a$.



Surface 1

$$\Phi = -E_I A + E_{III} A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\rho V}{\varepsilon_0} \quad \text{Gauss}$$

$$= \frac{2aA\rho}{\varepsilon_0}$$

Symmetry $\vec{E}_I = -\vec{E}_{III}$

$$-E_I A + (-E_{III})A = \frac{2aA\rho}{\varepsilon_0}$$

$$-2E_I A =$$

$$E_{I} = \frac{-a\rho}{\varepsilon_0} \qquad E_{III} = \frac{a\rho}{\varepsilon_0}$$

<u>Surface 2</u>

$$Q_{enc} = \rho A(x+a)$$

<u>Gauss</u>

$$-E_{I}A + E_{II}A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\rho A(x+a)}{\varepsilon_0}$$

$$E_{II}(x) = E_{I} + \frac{\rho}{\varepsilon_0}(x+a)$$

$$= \frac{-a\rho}{\varepsilon_0} + \rho/\varepsilon_0 (x+a)$$

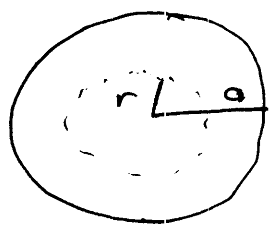
$$= \frac{\rho x}{\varepsilon_0}$$

$$\vec{E} = \begin{cases} -\dfrac{a\rho}{\varepsilon_0} & x < -a \\[2ex] \dfrac{\rho x}{\varepsilon_0} & -a < x < a \\[2ex] \dfrac{a\rho}{\varepsilon_0} & x > a \end{cases}$$

Ex Consider a spherically symmetric system with uniform volume charge density $\rho = \gamma r$ for $r < a$ and $\rho = 0$ for $r > a$.

Compute field everywhere.

Sln



Select a spherical Gaussian surface of radius r

For $r < a$

$$Q_{enc} = \int_{\text{Gaussian surface}} \rho \, d\tau$$

$$= \int_0^r dr \int_0^{\pi} r\,d\theta \int_0^{2\pi} r\sin\theta\, d\phi \; \rho(r)$$

$$= \int_0^r 4\pi r^2 \rho(r)\, dr$$

$$= \int_0^r 4\pi r^2 \gamma r \, dr = 4\pi\gamma \int_0^r r^3\, dr$$

$$Q_{enc} = 4\pi\gamma \frac{r^4}{4}\Big|_0^r = \pi\gamma r^4$$

<u>Gauss' Law</u>

$$\Phi = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} = \frac{\pi\gamma r^4}{\varepsilon_0}$$

$$E = \frac{\gamma r^2}{4\varepsilon_0}$$

<u>For $r > a$</u> All charge enclosed

$$Q_{enc} = \int \rho \, d\tau' = \frac{\pi\gamma a^4}{}$$

$$4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} = \frac{\pi\gamma a^4}{\varepsilon_0} \equiv Q_T$$

$$E = \frac{\pi\gamma a^4}{4\pi\varepsilon_0 r^2} = \frac{Q_T}{4\pi\varepsilon_0 r^2}$$

Note, outside all charge the field is that of a point charge.

Inside all charge, the field is zero.

$$\vec{E} = \begin{cases} \dfrac{\gamma r^2}{4\varepsilon_0}\,\hat{r} & r < a \\[2ex] \dfrac{\gamma a^4}{4\varepsilon_0 r^2}\,\hat{r} & r < a \end{cases}$$

All vectors measured from origin.