Vector Fields

The vanishing of the second vector derivatives

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \quad \text{and} \quad \nabla \times \nabla f = 0$$

will allow a dramatic simplification of our mathematically treatment of electromagnetic fields.

<u>Helmholtz Thm</u> - If a vector field goes to zero at infinity, the field is uniquely determined by its divergence and curl.

$\Rightarrow$ This is good because what Maxwell's eqns give us is the div and curl of $\vec{E}$ and $\vec{B}$.

~~By Helmholtz thm, if the curl of a function~~
~~is zero then the~~

Curl-Free Fields If $\nabla \times \vec{F} = 0$, then

- $\int_{\vec{a} \to \vec{b}} \vec{F} \cdot d\vec{l}$ is independent of path
- $\oint \vec{F} \cdot d\vec{l} = 0$ for any closed loop.
- $\vec{F} = -\nabla V$
- The field is irrotational.

Divergence-less Fields If $\nabla \cdot \vec{F} = 0$,

- $\vec{F} = \nabla \times \vec{A}$ for some $\vec{A}$
- $\int_S \vec{F} \cdot d\vec{a}$ is independent of surface for any C.
- $\oint \vec{F} \cdot d\vec{a} = 0$ for all closed surfaces.
- The field is solenoidal

Back to Maxwell

Gauss

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V \nabla \cdot \vec{E} \, d\tau = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V \rho \, d\tau$$

$$\underbrace{\oint_S \vec{E} \cdot d\vec{a} = \int_V \nabla \cdot \vec{E} \, d\tau}_{\text{divergence thm}}$$

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \qquad \text{Gauss' Law}$$

Gauss for Magnetism

$$\oint_S \vec{B} \cdot d\vec{a} = 0 = \int_V (\nabla \cdot \vec{B}) \, d\tau = 0$$

for all V

$$\nabla \cdot \vec{B} = 0$$

Faraday

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a}$$

$$= \int_S (\nabla \times \vec{E}) \cdot d\vec{a} \qquad \text{Stokes}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Ampere

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{a}$$

$$= \mu_0 \int_S \vec{J}\cdot d\vec{a} + \mu_0 \varepsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{a}$$

Stokes

$$\oint (\nabla\times\vec{B})\cdot d\vec{a} =$$

$$\nabla\times\vec{B} = \mu_0 \vec{J} + \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The magnetic field is solenoidal $\nabla\cdot\vec{B} = 0$ the vector potential so $\exists$ a function $\vec{A}$ called

s.t. $\vec{B} = \nabla\times\vec{A}$.

If $\vec{B}$ is constant, the electric field is curl free, $\nabla\times\vec{E} = 0 \implies \exists$ a function V s.t. $\vec{E} = -\nabla V$. V is the scalar potential.

If B is changing,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{A}$$

$$= \nabla \times \left(-\frac{\partial \vec{A}}{\partial t} \right)$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$