

Homework 1

Due Thursday 1/21/2010 - at beginning of class

Reading Assignment - Chapter 1

Griffiths' Problems

1.5

1.6

1.11

1.13

1.15

1.18

1.24

1.25

Problem E.1.1 Calculate ∇V where $V = \vec{p} \cdot \vec{r}/r^3$ where \vec{p} is a constant vector. Note this is the potential of a point dipole.

Problem E.1.2 Consider the vector field $\vec{E} = \gamma \hat{z} \times \vec{r}$ where γ is a constant. Sketch the field. Compute the line integral of the field around a circle of radius R in the $x - y$ plane by direct integration. Compute the same integral using Stoke's Thm.

Problem E.1.3 Consider the vector field $\vec{E} = \gamma \vec{r}$ where γ is a constant. Sketch the field. Compute the flux, $\oint \vec{E} \cdot d\vec{a}$, out of the cube $0 < x < 1$, $0 < y < 1$, and $0 < z < 1$ by direct integration. Compute the flux using the Divergence Thm.

Problem E.1.4 Consider the function $f = x^2 + y^2 + z^2$. Compute the gradient in both Cartesian and spherical coordinates. Show the answers agree.

1.5

Show $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \hat{x}(B_y C_z - B_z C_y)$$

$$- \hat{y}(B_x C_z - B_z C_x)$$

$$+ \hat{z}(B_x C_y - B_y C_x)$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix}$$

x-component

$$A_y (B_x C_y - B_y C_x) - A_z (B_z C_x - B_x C_z)$$

① ③ ④ ②

Work out $\vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})$

$$= \vec{B} (A_x C_x + A_y C_y + A_z C_z) - \vec{C} (A_x B_x + A_y B_y + A_z B_z)$$

x-component

$$\begin{aligned} & B_x A_x C_x + B_x A_y C_y + B_x A_z C_z \\ & - C_x A_x B_x - C_x A_y B_y - C_x A_z B_z \end{aligned}$$

① ② ③ ④

Underlined terms cancel, ① matches terms with x-component of $\vec{A} \times (\vec{B} \times \vec{C})$

That was boring. I worked out the full result with Maple as follows.

> with(LinearAlgebra);

[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, (1)
BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension,
ColumnOperation, ColumnSpace, CompanionMatrix, ConditionNumber, ConstantMatrix,
ConstantVector, Copy, CreatePermutation, CrossProduct, DeleteColumn, DeleteRow,
Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct,
EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute,
FrobeniusForm, GaussianElimination, GenerateEquations, GenerateMatrix,
GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix,
HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix,
IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary,
JordanBlockMatrix, JordanForm, LA_Main, LUdecomposition, LeastSquares, LinearSolve,
Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse,
MatrixMatrixMultiply, MatrixNorm, MatrixPower, MatrixScalarMultiply,
MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm,
Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, QRdecomposition,
RandomMatrix, RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm,
Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector,
SchurForm, SingularValues, SmithForm, SubMatrix, SubVector, SumBasis, SylvesterMatrix,
ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix,
VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply,
ZeroMatrix, ZeroVector, Zip]

> assume(Ax, real);

> assume(Ay, real);

> assume(Az, real);

> assume(Bx, real);

> assume(By, real);

> assume(Bz, real);

> assume(Cx, real);

> assume(Cy, real);

> assume(Cz, real);

> A := < Ax, Ay, Az >;

$$A := \begin{bmatrix} Ax\sim \\ Ay\sim \\ Az\sim \end{bmatrix} \quad (2)$$

> B := < Bx, By, Bz >;

$$B := \begin{bmatrix} Bx\sim \\ By\sim \\ Bz\sim \end{bmatrix} \quad (3)$$

> C := < Cx, Cy, Cz >;

$$C := \begin{bmatrix} Cx \\ Cy \\ Cz \end{bmatrix} \quad (4)$$

> left := CrossProduct(A, CrossProduct(B, C));

$$left := \begin{bmatrix} Ay (Bx Cy - By Cx) - Az (Bz Cx - Bx Cz) \\ Az (By Cz - Bz Cy) - Ax (Bx Cy - By Cx) \\ Ax (Bz Cx - Bx Cz) - Ay (By Cz - Bz Cy) \end{bmatrix} \quad (5)$$

> right := B · DotProduct(A, C) - C · DotProduct(A, B);

$$right := \begin{bmatrix} (Ax Cx + Ay Cy + Az Cz) Bx - (Ax Bx + Ay By + Az Bz) Cx \\ (Ax Cx + Ay Cy + Az Cz) By - (Ax Bx + Ay By + Az Bz) Cy \\ (Ax Cx + Ay Cy + Az Cz) Bz - (Ax Bx + Ay By + Az Bz) Cz \end{bmatrix} \quad (6)$$

> left - right;

$$\begin{bmatrix} \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} Ay (Bx Cy - By Cx) - Az (Bz Cx - Bx Cz) - (Ax Cx + Ay Cy + Az Cz) Bx + (Ax Bx + Ay By + Az Bz) Cx, [\\ Az (By Cz - Bz Cy) - Ax (Bx Cy - By Cx) - (Ax Cx + Ay Cy + Az Cz) By + (Ax Bx + Ay By + Az Bz) Cy, [\\ Ax (Bz Cx - Bx Cz) - Ay (By Cz - Bz Cy) - (Ax Cx + Ay Cy + Az Cz) Bz + (Ax Bx + Ay By + Az Bz) Cz] \end{bmatrix}$$

> simplify(%);

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

>

1.6 We have already performed the first cross-product in the previous problem, so nothing much is gained by the tedium of working out the next two. I show the solution in Maple on the next page.

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) = -\vec{C} \times (\vec{A} \times \vec{B})$$
$$= (\vec{A} \times \vec{B}) \times \vec{C}$$

so $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \times \vec{C}$
iff $\vec{B} \times (\vec{C} \times \vec{A}) = 0$

>

> with(LinearAlgebra);

[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations, GenerateMatrix, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, LA_Main, LUDecomposition, LeastSquares, LinearSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply, MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SubMatrix, SubVector, SumBasis, SylvesterMatrix, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip]

> assume(Ax, real);

> assume(Ay, real);

> assume(Az, real);

> assume(Bx, real);

> assume(By, real);

> assume(Bz, real);

> assume(Cx, real);

> assume(Cy, real);

> assume(Cz, real);

> A := < Ax, Ay, Az >;

$$A := \begin{bmatrix} Ax\sim \\ Ay\sim \\ Az\sim \end{bmatrix} \quad (2)$$

> B := < Bx, By, Bz >;

(3)

$$B := \begin{bmatrix} Bx\sim \\ By\sim \\ Bz\sim \end{bmatrix} \quad (3)$$

> C := < Cx, Cy, Cz >;

$$C := \begin{bmatrix} Cx\sim \\ Cy\sim \\ Cz\sim \end{bmatrix} \quad (4)$$

> CrossProduct(A, CrossProduct(B, C)) + CrossProduct(B, CrossProduct(C, A))
+ CrossProduct(C, CrossProduct(A, B))

$$\left[\right] \quad (5)$$

$$\begin{aligned} & Ay\sim (Bx\sim Cy\sim - By\sim Cx\sim) - Az\sim (Bz\sim Cx\sim - Bx\sim Cz\sim) \\ & + By\sim (Cx\sim Ay\sim - Cy\sim Ax\sim) - Bz\sim (Cz\sim Ax\sim - Cx\sim Az\sim) \\ & + Cy\sim (Ax\sim By\sim - Ay\sim Bx\sim) - Cz\sim (Az\sim Bx\sim - Ax\sim Bz\sim), [\\ & Az\sim (By\sim Cz\sim - Bz\sim Cy\sim) - Ax\sim (Bx\sim Cy\sim - By\sim Cx\sim) \\ & + Bz\sim (Cy\sim Az\sim - Cz\sim Ay\sim) - Bx\sim (Cx\sim Ay\sim - Cy\sim Ax\sim) \\ & + Cz\sim (Ay\sim Bz\sim - Az\sim By\sim) - Cx\sim (Ax\sim By\sim - Ay\sim Bx\sim)], [\\ & Ax\sim (Bz\sim Cx\sim - Bx\sim Cz\sim) - Ay\sim (By\sim Cz\sim - Bz\sim Cy\sim) \\ & + Bx\sim (Cz\sim Ax\sim - Cx\sim Az\sim) - By\sim (Cy\sim Az\sim - Cz\sim Ay\sim) \\ & + Cx\sim (Az\sim Bx\sim - Ax\sim Bz\sim) - Cy\sim (Ay\sim Bz\sim - Az\sim By\sim)] \end{aligned}$$

> simplify(%);

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

>

1.11

$$(a) \quad \nabla(x^2 + y^3 + z^4)$$

$$= (2x, 3y^2, 4z^3)$$

$$(b) \quad \nabla(x^2 y^3 z^4) = (2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3)$$

$$(c) \quad \nabla(e^x \sin y \ln z)$$

$$= \left(e^x \sin y \ln z, e^x \cos y \ln z, \frac{e^x \sin y}{z} \right)$$

1.13

$$\vec{r}'' = \vec{r} - \vec{r}'$$

$$= (x - x', y - y', z - z')$$

(a)

$$r''^2 = \vec{r}'' \cdot \vec{r}'' = (x - x')^2 + (y - y')^2 + (z - z')^2$$

$$\nabla r''^2 = \left(\frac{\partial}{\partial x} r''^2, \frac{\partial}{\partial y} r''^2, \frac{\partial}{\partial z} r''^2 \right)$$

$$= (2(x - x'), 2(y - y'), 2(z - z'))$$

$$= 2 \vec{r}''$$

$$(b) \nabla \left(\frac{1}{r''} \right) = \nabla \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

$$= -\frac{1}{2} \frac{1}{((x - x')^2 + (y - y')^2 + (z - z')^2)^{3/2}} \times$$

$$\left(2(x - x'), 2(y - y'), 2(z - z') \right)$$

$$\nabla \left(\frac{1}{r''} \right) = - \frac{\vec{r}''}{r''^3} = - \frac{\hat{r}''}{r''^2}$$

$$\begin{aligned} (c) \quad \nabla r''^n &= \nabla \left(\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \right)^n \\ &= \left(\frac{n}{2} \right) \left((x-x')^2 + (y-y')^2 + (z-z')^2 \right)^{\left(\frac{n}{2} - 1 \right)} \\ &\quad \cdot 2 \vec{r}'' \\ &= n r''(n-2) r'' \hat{r}'' \\ &= n r''(n-1) \hat{r}'' \end{aligned}$$

1.15

$$(a) \quad \nabla \cdot (x^2, 3xz^2, -2xz)$$

$$= 2x + 0 - 2x = 0$$

$$(b) \quad \nabla \cdot (xy, 2yz, 3zx)$$

$$= y + 2z + 3x$$

$$(c) \quad \nabla \cdot (y^2, 2xy + z^2, 2yz)$$

$$= 0 + 2x + 2y$$

1.18

$$(a) \quad \nabla \times \vec{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix}$$

$$= \hat{x} (0 - 6xz)$$

$$- \hat{y} (-2x - 0)$$

$$+ \hat{z} (3z^2 - 0)$$

$$= (-6xz, +2x, +3z^2)$$

$$(b) \quad \nabla \times \vec{v}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix}$$

$$\nabla \times \vec{V}_b = \hat{x}(0 - 2z) \\ - \hat{y}(3z - 0) \\ + \hat{z}(0 - x)$$

$$= (-2z, -3z, -x)$$

$$(c) \quad \nabla \times \vec{V}_c = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy + z^2) & 2yz \end{vmatrix}$$

$$= \hat{x}(2z - 2z)$$

$$- \hat{y}(0 - 0)$$

$$+ \hat{z}(2y - 2y) = 0$$

1.24 (a) Product rule iv

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\vec{A} = (x, 2y, 3z) \quad \vec{B} = (3x, -2y, 0)$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & 2y & 3z \\ 3x & -2y & 0 \end{vmatrix}$$

$$= \hat{x} (0 + 6xz)$$

$$- \hat{y} (-9yz)$$

$$+ \hat{z} (-2x^2 - 6y^2)$$

$$= (6xz, +9yz, -2x^2 - 6y^2)$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = 6z + 9z = 15z$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix}$$

$$= \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(0-0) = 0$$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix}$$

$$= \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(-2-3) = -5\hat{z}$$

$$\vec{B} \cdot (\nabla \times \vec{A}) = 0 \quad \vec{A} \cdot (\nabla \times \vec{B}) = -15z$$

$$\vec{B} \cdot (\nabla \times \vec{A}) + \vec{A} \cdot (\nabla \times \vec{B}) = -15z \quad \checkmark$$

(b) Product Rule ii

$$\begin{aligned}\nabla(\vec{A} \cdot \vec{B}) &= \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) \\ &\quad + (\vec{A} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A}\end{aligned}$$

$$\vec{A} \cdot \nabla = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + 3z \frac{\partial}{\partial z}$$

$$(\vec{A} \cdot \nabla) \vec{B} = (6y, -2x, 0)$$

$$\begin{aligned}(\vec{B} \cdot \nabla) \vec{A} &= \left(3y \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y} \right) (x, 2y, 3z) \\ &= (3y, -4x, 0)\end{aligned}$$

We calculated $\nabla \times \vec{A}$ and $\nabla \times \vec{B}$ in the previous problem

$$\vec{A} \times (\nabla \times \vec{B}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix}$$

$$= \hat{x} (-10y - 0)$$

$$- \hat{y} (-5x - 0)$$

$$+ \hat{z} (0 - 0)$$

$$= (-10y, 5x, 0)$$

$$\vec{B} \times (\nabla \times \vec{A}) = 0$$

$$\vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A}$$

$$= (-10y, 5x, 0) + 0 + (6y, -2x, 0)$$

$$+ (3y, -4x, 0)$$

$$= (-y, -x, 0)$$

$$\nabla(\vec{A} \cdot \vec{B}) = \nabla(3xy - 4xy)$$

$$= -\nabla(xy) = (-x, -x, 0) \quad \checkmark$$

(c) Product Rule vi

$$\begin{aligned} \nabla \times (\vec{A} \times \vec{B}) &= (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} \\ &+ \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) \end{aligned}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{A} = (1+2+3) = 6$$

$$\nabla \times (\vec{A} \times \vec{B}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz & 9yz & -2x^2 - 6y^2 \end{vmatrix}$$

$$= \hat{x}(-12y - 9y)$$

$$- \hat{y}(-4x - 6x)$$

$$+ \hat{z}(0 - 0) = (-21y, +10x, 0)$$

$$\vec{A} (\nabla \cdot \vec{B}) = 0$$

$$\vec{B} (\nabla \cdot \vec{A}) = (18y, -12x, 0)$$

$$(\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A})$$

$$= (3y, -4x, 0) - (6y, -2x, 0) + 0 - (18y, -12x, 0)$$

$$= (-21y, 10x, 0) \quad \checkmark$$

1.25

$$(d) \quad \nabla^2 T_a = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (x^2 + 2xy + 3z + 4)$$
$$= 2$$

$$(b) \quad \nabla^2 (\sin x \sin y \sin z)$$
$$= -3 \sin x \sin y \sin z \quad (\text{back to QM})$$

$$(c) \quad \nabla^2 (e^{-5x} \sin 4y \cos 3z)$$
$$= \left(\begin{array}{l} 25 e^{-5x} \sin 4y \cos 3z \\ -16 e^{-5x} \sin 4y \cos 3z \\ -9 e^{-5x} \sin 4y \cos 3z \end{array} \right) = 0$$

$$(d) \quad \nabla^2 \vec{v} = \left(\nabla^2 x^2, \nabla^2 3xz^2, -\nabla^2 zxz \right)$$
$$= (2, 6x, 0)$$

(E.1.1)

$$V = \frac{\vec{P} \cdot \vec{r}}{r^3}$$
$$= \frac{1}{r^3} (P_x x + P_y y + P_z z)$$

Product Rule i

$$\nabla fg = g \nabla f + f \nabla g$$

$$f = 1/r^3$$
$$g = \vec{P} \cdot \vec{r}$$

$$\nabla (P_x x + P_y y + P_z z) = \vec{P}$$

$$\nabla \frac{1}{r^3} = -3 \frac{1}{r^4} \hat{r} \quad (\text{spherical})$$

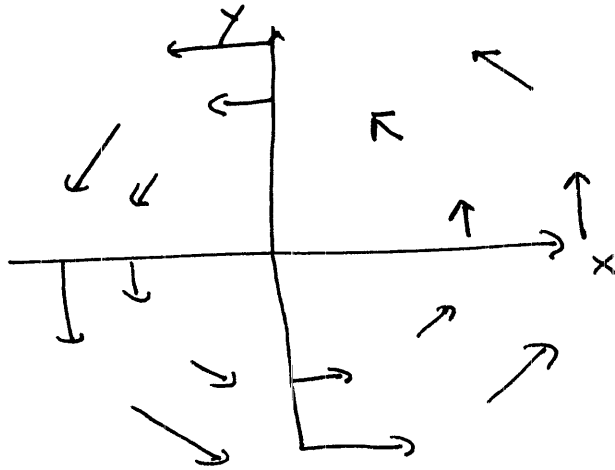
$$\nabla V = (\vec{P} \cdot \vec{r}) \left(-3 \frac{1}{r^4} \hat{r}\right) + \frac{1}{r^3} \vec{P}$$

$$= (\vec{P} \cdot \hat{r}) \left(-3 \frac{1}{r^3} \hat{r}\right) + \frac{1}{r^3} \vec{P}$$

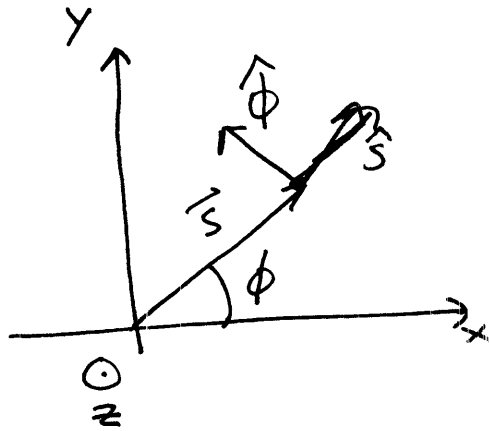
$$= \frac{1}{r^3} \left(\vec{P} - 3(\vec{P} \cdot \hat{r}) \hat{r} \right)$$

E.1.2

$$\vec{E} = \gamma \hat{z} \times \vec{r}$$



In x-y plane $\vec{r} = \vec{s}$ and we can use cylindrical coordinates.



$$\hat{s} \times \hat{\phi} = \hat{z}$$

$$\hat{z} \times \hat{s} = \hat{\phi}$$

using the coordinate system drawn above.

$$\begin{aligned}
 \text{So } \vec{E} &= \gamma \hat{z} \times \vec{r} = \gamma \hat{z} \times \vec{s} \\
 &= \gamma s \hat{z} \times \hat{s} = \gamma s \hat{\phi}
 \end{aligned}$$

Integrate around circle of radius R

$$\oint_C \vec{E} \cdot d\vec{\ell} = \oint_C (\gamma s \hat{\phi}) \cdot (s d\phi \hat{\phi}) \quad \begin{array}{l} d\vec{\ell} = s d\phi \hat{\phi} \\ s = R \end{array}$$

$$= \gamma R^2 \oint_C d\phi$$

$$= 2\pi \gamma R^2$$

Stoke's Thm

Compute ~~the curl~~ Curl

$$\hat{z} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix}$$

$$= \hat{x}(0 - y) - \hat{y}(0 - x) + \hat{z}(0 - 0)$$

$$= -y \hat{x} + x \hat{y}$$

Compute Curl

$$\nabla \times \vec{E} = \gamma \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= \gamma \left[\hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(1+1) \right]$$

$$= 2\gamma \hat{z}$$

Integrate over surface of radius R

$$d\vec{a} = \hat{z} d\vec{a} = \cancel{\hat{z} r dr d\phi}$$

$$= \hat{z} (ds) (d\phi s)$$

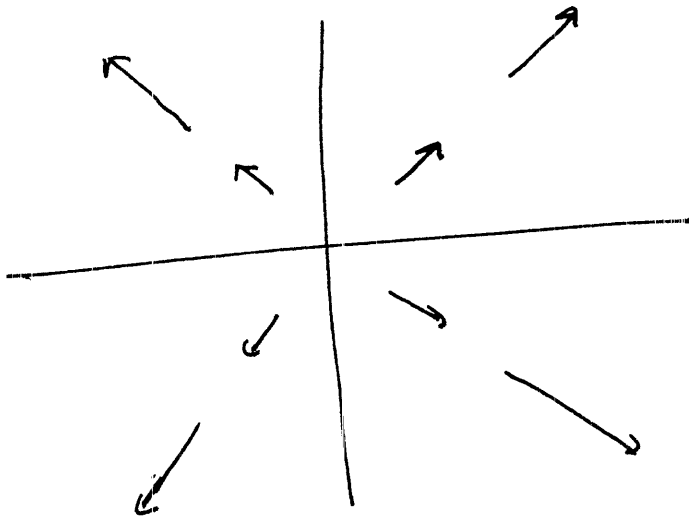
$$\int_S (\nabla \times \vec{E}) \cdot d\vec{a} = \int_S (2\gamma \hat{z}) \cdot (\hat{z} s ds d\phi)$$

$$= 2\gamma \int_0^{2\pi} d\phi \int_0^R s ds$$

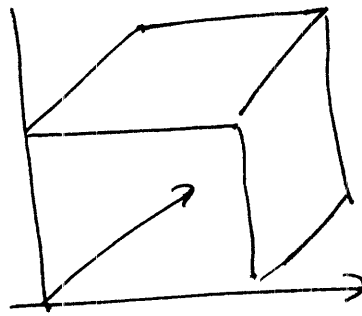
$$= 2\gamma \frac{R^2}{2} \int_0^{2\pi} d\phi = 2\pi \gamma R^2 \quad \checkmark$$

E.1.3

$$\vec{E} = \gamma \vec{r}$$



Φ out of $0 < x < 1, 0 < y < 1, 0 < z < 1$
cube



$\vec{E} \perp \hat{n}$ for the sides in the coordinate planes.

The other 3 sides are symmetrically placed,

$$\text{so } \Phi = 3\Phi_{\text{top}}$$

$$\underline{\Phi}_{\text{top}} = \int_{\text{top}} \vec{E} \cdot d\vec{\sigma}$$

$$d\vec{\sigma} = \hat{z} dx dy$$

$$\vec{E} = \gamma(x, y, z)$$

$$\vec{E} \cdot d\vec{\sigma} = \gamma z dx dy$$

$$\underline{\Phi}_{\text{top}} = \int \vec{E} \cdot d\vec{\sigma} = \int_0^1 dx \int_0^1 dy z \gamma \quad z=1$$

$$= 1\gamma$$

$$\underline{\Phi} = 3 \underline{\Phi}_{\text{top}} = 3\gamma$$

Divergence

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \cdot \gamma r) + \text{zero stuff}$$

$$= 3\gamma$$

$$\vec{E} = \gamma \vec{r} = \gamma r \hat{r}$$

$$\int \nabla \cdot \vec{E} \, d\tau = \int_0^1 dx \int_0^1 dy \int_0^1 dz \, 3\gamma$$

$$= 3\gamma \quad \checkmark$$

E.1.4

$$f = x^2 + y^2 + z^2 = r^2$$

Cartesian

$$\begin{aligned}\nabla f &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2) \\ &= (2x, 2y, 2z) \\ &= 2\vec{r}\end{aligned}$$

Spherical

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial r} \hat{r} + \text{zero stuff} \\ &= 2r\hat{r} = 2\vec{r} \checkmark\end{aligned}$$