# Homework 3

Due Wednesday 2/10/2010 - at 5:00pm

Reading Assignment - Chapter 3 Except Section 3.4

## Griffiths' Problems

**3.8**

**3.9**

**3.12**

**3.18** Hint - Use trig identities the write the potential in terms of the Legendre polynomials.

**3.22** Work only the potential outside and calculate only the first 4 terms up to $P_3$ explicitly.

## Additional Problems

**E.3.1** Finish the conducting channel I proposed in class. Find the potential inside a rectangular channel with $V(x, y, 0) = V_0$, $V(x, 0, z) = 0$, $V(x, a, z) = 0$, $V(0, y, z) = 0$, $V(a, y, z) = 0$. This channel has its long sides grounded and its end at $V_0$.

**E.3.2** The potential at the surface of an infinite cylinder of radius $a$ is $V(a, \phi, z) = V_0 \cos(3\phi)$. Find the potential both inside and outside the cylinder. Find the field inside and outside and the surface charge density on the cylinder.

**E.3.3** Find the potential in the region where $x > 0$ and $y > 0$. The $y - z$ plane is held at potential $V_0$ and the $x - z$ plane is grounded. Hint, look at the trivial solutions.

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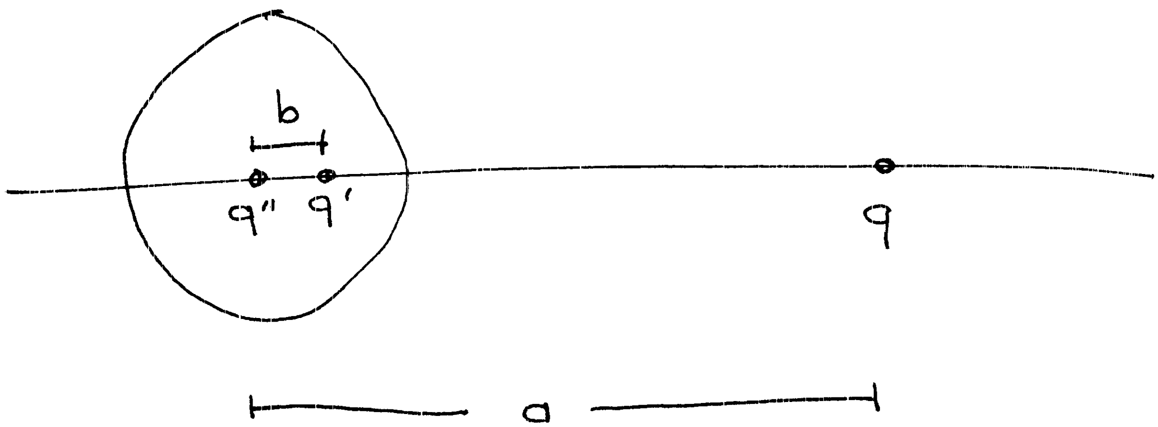
(3.8) Force of attraction point charge q and neutral conducting sphere.




To bring the surface of the sphere to zero potential use an image charge

$$q' = -\frac{R}{a} q$$

$$\text{at } b = \frac{R^2}{a}$$

This leaves the sphere with charge $q'$. To return the sphere to $Q = 0$. Place a charge $q'' = 0 - q'$ at the origin.

$$F = \frac{kqq''}{a^2} + \frac{kqq'}{(a-b)^2}$$

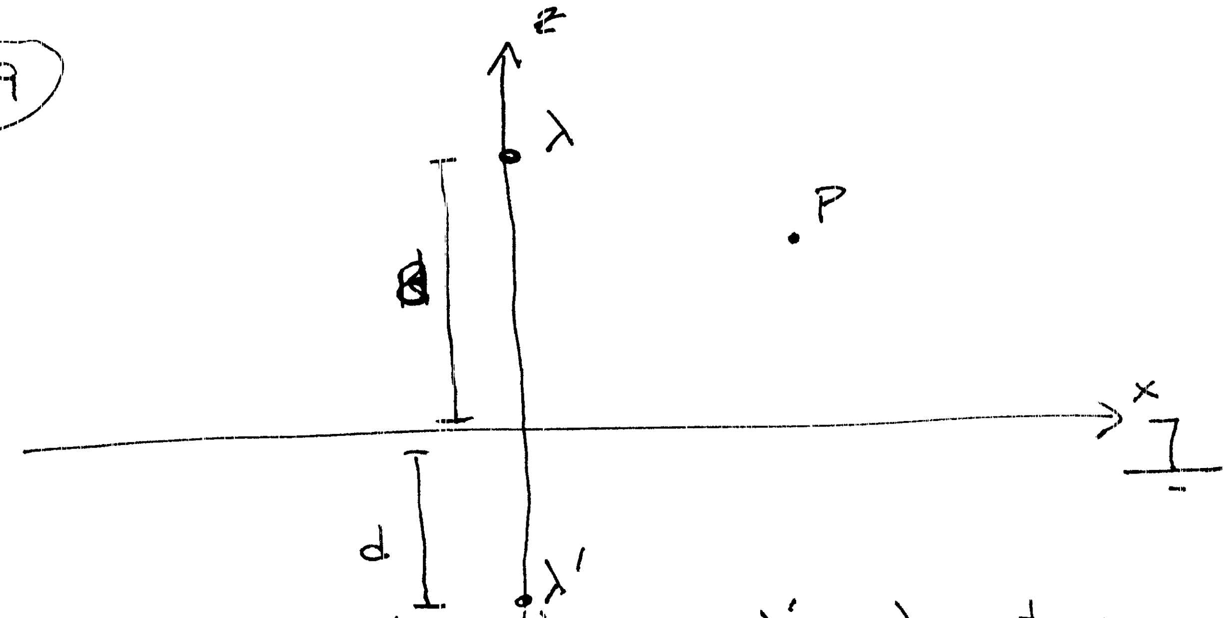
$$= kqq' \left( \frac{1}{(a-b)^2} - \frac{1}{a^2} \right)$$

$$= -\frac{kq^2R}{a} \left( \frac{1}{(a - R^2/a)^2} - \frac{1}{a^2} \right)$$

Now the first part of the problem. To leave the surface at $V_0$ place a charge $q''$ in the center. Choose $q''$ so that

$$V_0 = \frac{kq''}{R}$$

3.9




Sln  Use image line charge, $\lambda' = -\lambda$ at a distance d below the plane.

The field of an infinite line charge is

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

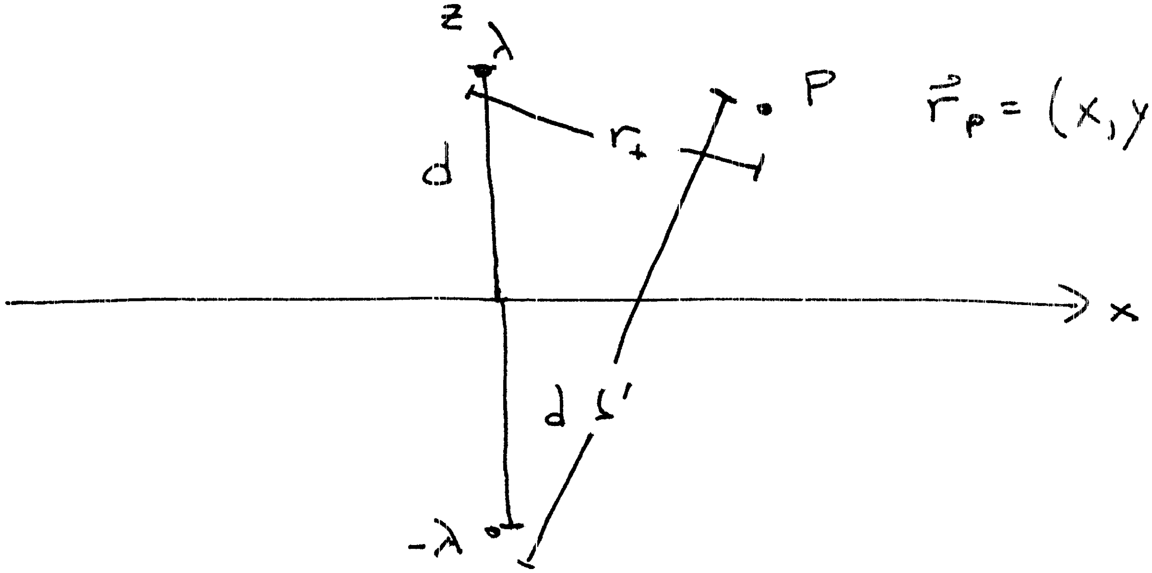
so the potential is

$$V = -\int E\,dr = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r) + C$$

Make the choice, $V(0,0,0) = 0 \Rightarrow -\frac{\lambda}{2\pi\varepsilon_0}\ln(d) + C = 0$

$$C = \frac{\lambda}{2\pi\varepsilon_0}\ln(d)$$

$$V(r) = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r/d)$$




$\vec{r}_p = (x, y, z)$

$r_+ = \sqrt{x^2 + (z-d)^2}$ $\qquad$ $r_- = \sqrt{x^2 + (z+d)^2}$

$$V_p = \frac{-\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_+}{d}\right) + \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_-}{d}\right)$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_-}{r_+}\right) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{\sqrt{x^2+(z+d)^2}}{\sqrt{x^2+(z-d)^2}}\right)$$

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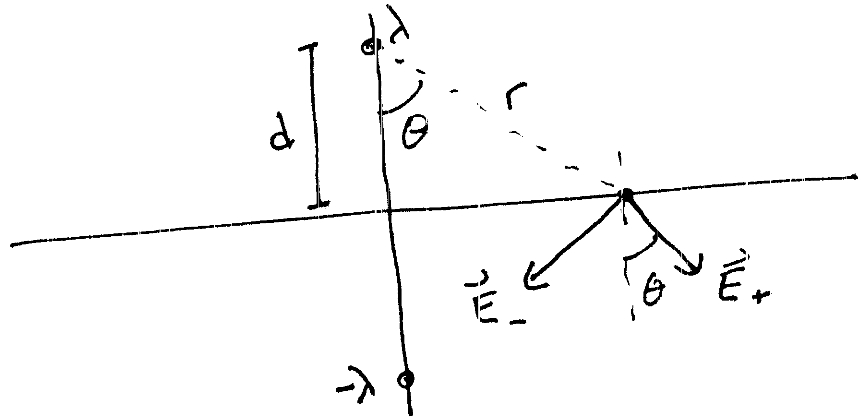
(b) The electric field at a point on the surface



The $x, y$ components of the field cancel

$$\vec{E}_{plane} = \vec{E}_+ + \vec{E}_- = 2\,|E_+|\cos\theta$$

$$= \frac{2\lambda}{2\pi\varepsilon_0 r} \cdot \frac{d}{r}$$

$$= \frac{\lambda d}{\pi\varepsilon_0} \frac{1}{r^2}$$

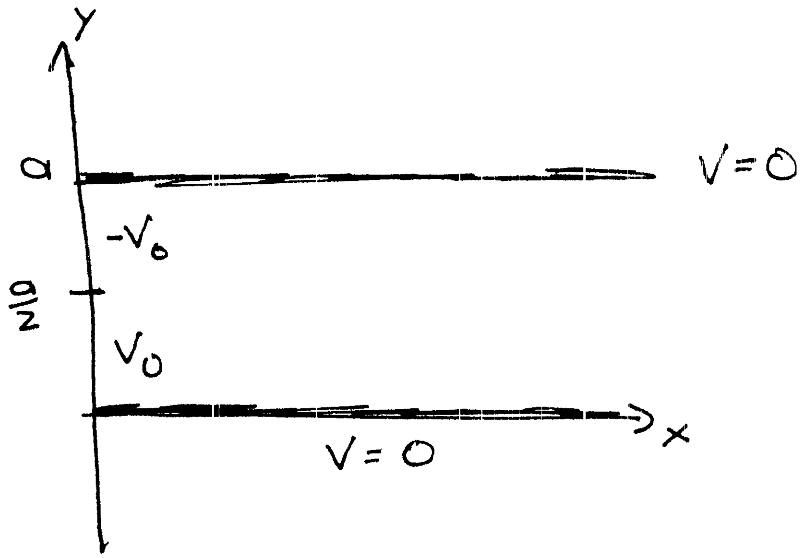
Use Gaussian pillbox at surface

$$\underline{\Phi} = \vec{E}\cdot\hat{n}\,A = \frac{\lambda d}{\pi\varepsilon_0}\frac{1}{r^2} A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\sigma = \frac{-\lambda d}{\pi r^2} = \frac{-\lambda d}{\pi (x^2 + d^2)}$$

Note, $\sigma$ must be negative

3.12




Two dimensional Laplace's Eqn

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Separate $\quad V = X(x)\, Y(y)$

$$\underbrace{\frac{1}{X(x)} \frac{d^2 X}{dx^2}}_{= +k^2} + \underbrace{\frac{1}{Y} \frac{d^2 Y}{dy^2}}_{= -k^2} = 0$$

Y eqn

$$\frac{d^2Y}{dy^2} + k^2 Y = 0$$

$Y = \sin kx, \ \cos kx$

For BC, $V(x,0,z) = 0$  $V(x,a,z) = 0$

chose sine solution and let $k_n = \frac{n\pi}{a}$.

X eqn

$$\frac{d^2X}{dx^2} - k^2 X = 0$$

$X = e^{kx}, \ e^{-kx}$

For the solution to be finite as $x \to \infty$,

$X(x) = e^{-k_n x}$

General Solution

$$V(x,y) = \sum_n A_n e^{-k_n x} \sin k_n y$$

Impose $x=0$ boundary condition

$$V(0,y) = \begin{cases} V_0 & 0<y<a/2 \\ -V_0 & a/2<y<a \end{cases}$$

$$= \sum_n A_n \sin k_n y$$

Multiply by $\sin k_m \cancel{x}$

$$V_0 \int_0^{a/2} \sin k_m y \, dy - V_0 \int_{a/2}^{a} \sin k_m y \, dy$$

$$= \sum_n A_n \underbrace{\int_0^a \sin k_m y \sin k_n y \, dy}_{\frac{a}{2}\delta_{nm}}$$

$$-\frac{V_0}{K_m}\cos K_m y\Big|_0^{a/2} + \frac{V_0}{K_m}\cos K_m y\Big|_{a/2}^{a}$$

$$= \frac{a}{2} A_m$$

$$= -\frac{V_0 a}{m\pi}\left(\cos\frac{m\pi}{2} - 1\right) + \frac{V_0 a}{m\pi}\left(\cos(m\pi) - \cos\frac{m\pi}{2}\right)$$

$$= \frac{V_0 a}{m\pi}\left(\cos m\pi - 2\cos\frac{m\pi}{2} + 1\right)$$

$= \dfrac{4 V_0 a}{m\pi}$ if $m = 2, 6, 10$ which I simply read from the solution

0 otherwise

Let's work on it a bit

$$\cos\frac{m\pi}{2} = \pm\sqrt{\frac{1+\cos m\pi}{2}}$$

No luck, try a table

| $m$ | $\cos m\pi$ | $-2\cos\frac{m\pi}{2}$ | 1 | Total |
|---|---|---|---|---|
| 1 | -1 | 0 | 1 | 0 |
| 2 | 1 | 2 | 1 | 4 |
| 3 | -1 | 0 | 1 | 0 |
| 4 | 1 | -2 | 1 | 0 |
| 5 | -1 | 0 | 1 | 0 |
| 6 | 1 | 2 | 1 | 4 |

So $A_m = \dfrac{8V_0}{m\pi}$ if $m = 2, 6, 10, 14 \ldots$

$$V(x,y) = \frac{8V_0}{\pi} \sum_n \frac{1}{n} e^{-k_n x} \sin k_n y$$

3.18 $V(R,\theta) = K\cos 3\theta$

<u>Trig identity</u> $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

<u>Legendre Polynomials</u>

$P_0 = 1 \qquad P_1 = \cos\theta \qquad P_2 = \frac{1}{2}(3\cos^2\theta - 1)$

$P_3 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

$$2P_3 = 5\cos^3\theta - 3P_1$$

$$\cos^3\theta = \frac{1}{5}(2P_3 + 3P_1)$$

$$\cos 3\theta = \frac{4}{5}(2P_3 + 3P_1) - 3P_1$$

$$= \frac{8}{5}P_3 - \frac{3}{5}P_1$$

$$V(R,0) = \frac{K}{5}(8P_3 - 3P_1)$$

<u>Outside</u> Discard $r^n$ terms

$$V_o(r,\theta) = \sum_n A_n r^{-(n+1)} P_n(\cos\theta)$$

Let $a = R$

$$V_o(a,\theta) = \frac{8k}{5} P_3 - \frac{3k}{5} P_1 = \sum_n A_n^o a^{-(n+1)} P_n$$

$$\frac{A_1^o}{a^2} = -\frac{3k}{5} \qquad \frac{A_3^o}{a^4} = \frac{8k}{5}$$

$$V_o(r,\theta) = -\frac{3ka^2}{5r^2} P_1(\cos\theta) + \frac{8ka^4}{5r^4} P_3(\cos\theta)$$

<u>Inside</u> discard $r^{-(n+1)}$ terms

$$V_i(a,\theta) = \sum_n A_n^i r^n P_n(\cos\theta)$$

$$= \frac{8k}{5} P_3 - \frac{3k}{5} P_1$$

$$a^3 A_3^i = \frac{8k}{5} \qquad a A_1^i = -\frac{3k}{5}$$

$$V_i(r,\theta) = -\frac{3k}{5a} r P_1(\cos\theta) + \frac{8k}{5a^3} r^3 P_3(\cos\theta)$$

Surface Charge Density

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = \frac{-\sigma}{\varepsilon_0}$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a = \frac{6ka^2}{5a^3} P_1 - \frac{32ka^4}{5a^5} P_3$$

$$\left.\frac{\partial V_i}{\partial r}\right|_a = \frac{-3k}{5a} P_1(\cos\theta) + \frac{24ka^2}{5a^3} P_3(\cos\theta)$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = \frac{k}{a}\left(\frac{9}{5} P_1 - \frac{56}{5} P_3\right) = \frac{-\sigma}{\varepsilon_0}$$

$$\sigma = \frac{k\varepsilon_0}{5a}\left(56 P_3 - 9 P_1\right)$$

3.22 Sphere with uniform surface charge density

$$\sigma(\theta) = \begin{cases} \sigma_0 & 0 \le \theta \le \pi/2 \\ -\sigma_0 & \frac{\pi}{2} < \theta < \pi \end{cases}$$

<u>Outside</u> (Discard $r^n$)

$$V_o(r,\theta) = \sum A_n^o r^{-(n+1)} P_n(\cos\theta)$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a = \sum -(n+1) A_n^o a^{-(n+2)} P_n(\cos\theta)$$

<u>Inside</u> (Discard $r^{-(n+1)}$)

$$V_i(r,\theta) = \sum A_n^i r^n P_n(\cos\theta)$$

$$\left.\frac{\partial V_i}{\partial r}\right|_a = \sum_n n A_n^i r^{n-1} P_n(\cos\theta)$$

## Potential Continuous at a

$$V_i(a,\theta) = V_o(a,\theta)$$

$$\sum_n A_n^i a^n P_n(\cos\theta) = \sum_n A_n^o a^{-(n+1)} P_n(\cos\theta)$$

By orthogonality, the series must be equal term by term.

$$A_n^i a^n = A_n^o a^{-(n+1)}$$

$$A_n^o = a^{2n+1} A_n^i$$

## Electrostatic Boundary Conditions at a

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = \frac{-\sigma}{\varepsilon_0}$$

$$\sum_n -(n+1) A_n^o a^{-(n+2)} P_n(\cos\theta) - \sum_n n A_n^i a^{n-1} P_n(\cos\theta) = -\frac{\sigma(x)}{\varepsilon_0}$$

$$-\sigma(x) = \sum_n P_n(\cos\theta)\Big[-A_n^o\, a^{-(n+2)}(n+1) - n A_n^i\, a^{(n-1)}\Big]\varepsilon_0$$

$$= \sum_n A_n^i \varepsilon_0 P_n(\cos\theta)\Big[-(n+1)\, a^{-(n+2)}\, a^{2n+1} - n\, a^{(n-1)}\Big]$$

$$= \sum_n \varepsilon_0 A_n^i P_n(\cos\theta)\, a^{(n-1)}\,[-2n-1]$$

$$\sigma(x) = \varepsilon_0 \sum_n (2n+1) A_n^i P_n(\cos\theta)\, a^{(n-1)}$$

<u>Multiply by $P_m$ and integrate</u>

$$\int_{-1}^{1} \sigma(\theta) P_m(\cos\theta)\, d(\cos\theta) = \varepsilon_0 \sum_n (2n+1) a^{n-1} A_n^i \,\times \underbrace{\int_{-1}^{1} P_m(\cos\theta) P_n(\cos\theta)\, d\cos\theta}_{\text{orthogonality condition}}$$

$$\text{orthogonality condition} = \frac{2}{2m+1}\delta_{nm}$$

$$\int_{-1}^{1} \sigma(\theta)\, P_m(\cos\theta)\, d(\cos\theta) = \varepsilon_0 \sum_n (2n+1) A_n^i a^{n-1} \cdot \frac{2}{2m+1} \delta_{nm}$$

$$= 2\, \varepsilon_0\, A_m^i\, a^{m-1}$$

<u>Work on the integral</u>

$$\int_{-1}^{1} \sigma(\theta)\, P_m(\cos\theta)\, d(\cos\theta) = -\int_{\pi}^{0} \sigma(\theta)\, P_m(\cos\theta) \sin\theta\, d\theta$$

$$= \int_{0}^{\pi} \sigma(\theta) \sin\theta\, P_m(\cos\theta)\, d\theta \equiv I_m$$

<u>Boundary Condition</u>

$$\sigma(\theta) = \begin{cases} \sigma_0 & 0 \leq \theta \leq \pi/2 \\ -\sigma_0 & \pi/2 \leq \theta \leq \pi \end{cases}$$

$$I_m = \sigma_0 \int_{0}^{\pi/2} P_m(\cos\theta) \sin\theta\, d\theta - \sigma_0 \int_{\pi/2}^{\pi} P_m(\cos\theta)\, d(\cos\theta)$$

$u = \cos\theta \qquad du = -\sin\theta\, d\theta$

$$I_m = -\sigma_0 \int_{-1}^{0} P_m(u)\,du + \sigma_0 \int_0^1 P_m(u)\,du$$

$$= \sigma_0 \int_0^1 P_m(u)\,du - \sigma_0 \int_{-1}^{0} P_m(u)\,du$$

Evidenty $I_m = 0$ for even functions, and

$$I_m = 2\sigma_0 \int_0^1 P_m(u)\,du$$

for odd functions.

~~$P_0$ even~~ $\cancel{I_0 = 2\sigma_0 \int_0^1 P_0\,du = 2\sigma_0 \int_0^1 du = 2\sigma_0}$

$P_0 = 1$, $I_0 = 0$ since even.

$P_1 = x$, $I_1 = 2\sigma_0 \int_0^1 x\,dx = \sigma_0$

$P_2 = \frac{1}{2}(3x^2 - 1)$, $I_2 = 0$ since even.

$P_3 = \frac{1}{2}(5x^3 - 3x)$ $\quad I_3 = \sigma_0 \int_0^1 (5x^3 - 3x)\,dx$

$$= \sigma_0\left(\frac{5}{4} - \frac{3}{2}\right) = \frac{-\sigma_0}{4}$$

$$I_0 = 0 \Rightarrow \boxed{A_0 = 0}$$

$$I_1 = \sigma_0 = 2\varepsilon_0 A_1^i a^{1-1}$$

$$\boxed{A_1^i = \frac{\sigma_0}{2\varepsilon_0}}$$

$$I_2 = 0 \Rightarrow \boxed{A_2 = 0}$$

$$I_3 = -\frac{\sigma_0}{4} = 2\varepsilon_0 A_3^i a^{3-1}$$

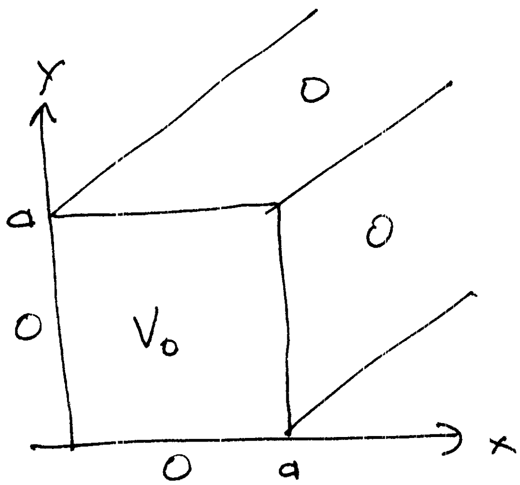
$$\boxed{A_3^i = -\frac{\sigma_0}{8\varepsilon_0 a^2}}$$

<u>Outside</u>

$$A_0^o = 0 \qquad A_1^o = a^3 A_1^i = \frac{\sigma_0 a^3}{2\varepsilon_0}$$

$$A_2^o = 0 \qquad A_3^o = a^7 A_3^i = -\frac{\sigma_0 a^5}{8\varepsilon_0}$$

## E.3.1




## Laplace's Eqn

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

## Separation

$$V = X\,Y\,Z$$

$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{= -k^2} + \underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{= -l^2} + \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{= w^2} = 0$$

$$w^2 = k^2 + l^2$$

To meet $V=0$ boundary conditions, choose sine solutions on let

$$k_n = \frac{n\pi}{a}$$

$$l_m = \frac{m\pi}{a}$$

yielding $w_{nm} = \sqrt{k_n^2 + l_m^2} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2}$

~~Thr~~ Throw away $e^{+wz}$ solution because it blows up at $\infty$.

<u>General Solution</u>

$$V(x,y,z) = \sum_{n,m} A_{nm} e^{-w_{nm} z} \sin k_n x \sin l_m y$$

<u>Impose $z=0$ boundary condition</u>

$$V(x,y,0) = V_0 = \sum A_{nm} \sin k_n x \sin l_m y$$

Orthogonality

$$\int_0^a \sin k_n x \sin k_m x \, dx = \frac{a}{2}\delta_{nm}$$

Multiply by $\sin k_i x \sin \ell_j y$ and integrate.

$$\int_0^a dx \int_0^a dy \; V_0 \sin k_i x \sin k_j y$$

$$= \sum_{nm} A_{nm} \underbrace{\left(\int_0^a dx \sin k_i x \sin k_n x\right)}_{\frac{a}{2}\delta_{in}} \underbrace{\left(\int_0^a \sin \ell_j y \sin \ell_m y \, dy\right)}_{\frac{a}{2}\delta_{jm}}$$

$$= \frac{a^2}{4} A_{ij}$$

$$= V_0 \left(\int_0^a dx \sin k_i x\right)\left(\int_0^a dy \sin \ell_j y\right)$$

$$= V_0 \left(\frac{1}{k_i}\left(-\cos k_i x\right)\Big|_0^a\right)\left(\frac{1}{\ell_j}\left(-\cos \ell_j y\right)\Big|_0^a\right)$$

$$= \frac{V_0}{k_i \ell_j} \underbrace{(1 - \cos i\pi)}_{\substack{= \\ 0 \ i \text{ even} \\ 2 \ i \text{ odd}}} \underbrace{(1 - \cos j\pi)}_{\substack{= \\ 0 \ j \text{ even} \\ 2 \ j \text{ odd}}}$$

$$= \frac{4V_0}{k_i \ell_j} \quad \text{if } i, j \text{ odd}, \quad 0 \text{ otherewise}$$

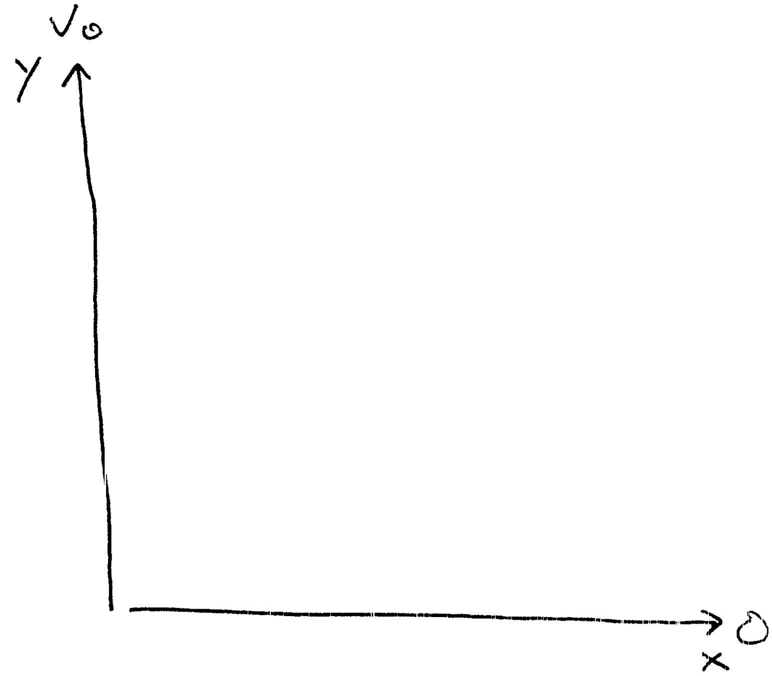
$$= \frac{4a^2 V_0}{\pi^2 i j} = \frac{a^2}{4} A_{ij}$$

$$A_{ij} = \frac{16 V_0}{\pi^2 i j}$$

<u>Solution</u>

$$V(x, y, z) = \frac{16 V_0}{\pi^2} \sum_{n,m} \frac{1}{nm} e^{-w_{nm} z} \sin k_n x \sin \ell_m y$$

E 3.3




Trivial Solution $V = C\phi + D$ satisfies' B.C.

so by uniqueness it is the solution.

$$V(0) = D = 0$$

$$V\left(\frac{\pi}{2}\right) = C\frac{\pi}{2} = V_0 \implies C = \frac{2V_0}{\pi}$$

$$V(s,\phi,z) = \frac{2V_0}{\pi}\phi$$

E. 3.2

Outside

$$V_o(s,\phi) = \sum_n A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi$$

By orthogonality, only $\cos 3\phi$ survives.

$$V_o(s,\phi) = \frac{A_3}{s^3} \cos 3\phi$$

B.C

$$V(a,\phi) = V_0 \cos 3\phi = \frac{A_3}{a^3} \cos 3\phi$$

$$A_3 = V_o a^3$$

$$V_o(s,\phi) = \frac{V_o a^3}{s^3} \cos 3\phi$$

## Inside

$$V_i(s,\phi) = \sum_n C_n s^n \cos n\phi + D_n s^n \sin n\phi$$

by orthogonality

$$V_i(s,\phi) = C_3 s^3 \cos 3\phi$$

## Boundary Conditions

$$V(a,\phi) = V_0 \cos 3\phi = C_3 a^3 \cos 3\phi$$

$$C_3 = V_0 / a^3$$

$$V_i(s,\phi) = \frac{V_0 s^3}{a^3} \cos 3\phi$$

Field

$$\nabla f = \hat{s}\frac{\partial f}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial f}{\partial \phi}$$

Field Inside

$$\vec{E}_i = -\nabla V_i = -\left(\frac{3V_0 s^2}{a^3}\cos 3\phi\,\hat{s} - \frac{3V_0 s^2}{a^3}\sin 3\phi\,\hat{\phi}\right)$$

Field Outside

$$\vec{E}_o = -\nabla V_o = -\left(-\frac{3V_0 a^3}{s^4}\cos 3\phi\,\hat{s} - \frac{3V_0 a^3}{s^4}\sin 3\phi\,\hat{\phi}\right)$$

or re-writing

$$\vec{E}_i = \frac{3V_0 s^2}{a^3}\left(-\cos 3\phi\,\hat{s} + \sin 3\phi\,\hat{\phi}\right)$$

$$\vec{E}_o = \frac{3V_0 a^3}{s^4}\left(\cos 3\phi\,\hat{s} + \sin 3\phi\,\hat{\phi}\right)$$

## Surface Charge

Use Gaussian pillbox at surface

$$\sigma = \varepsilon_0 \hat{n} \cdot \left( \vec{E}_o - \vec{E}_i \right) \qquad \hat{n} = \hat{s}$$

$$= \varepsilon_0 \left( \frac{3 V_0 a^3}{a^4} \cos 3\phi - \frac{3 V_0 a^2}{a^3} \left( -\cos 3\phi \right) \right)$$

$$= \frac{6 V_0 \varepsilon_0}{a} \cos 3\phi$$