Homework 5

Due Thursday 3/11/2010 - at 5:00pm

Reading Assignment - Chapter 5

Griffiths' Problems

5.4

5.5

5.9

5.11

5.16

5.22 You only need to find the potential. You do not need to take the curl to find the field.

5.23

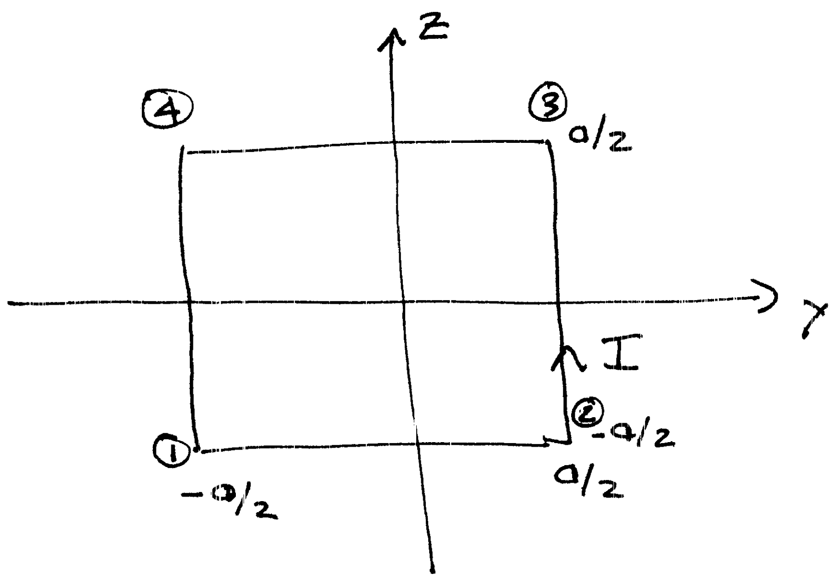
5.35

E1 A non-uniform current $\vec{J} = \gamma r^2 \hat{z}$ flows in the $\hat{z}$ direction in the region $a < s < b$. γ is a constant. Compute the magnetic field everywhere.

E2 Compute the vector potential at the center of a square element of current $\vec{K} = K_0 \hat{y}$ where the current extends from $x = -a$ to a and $y = -a$ to a in the $x - y$ plane. K_0 is a constant. A good problem to convince you guys to learn Maple.

5.4

$$\vec{B} = kz\hat{x}$$



$$\vec{F}_{12} = I\vec{L}\times\vec{B} = I(a\hat{y})\times\left(-\frac{a}{2}k\hat{x}\right)$$

$$= \frac{Ia^2}{2}k\hat{z}$$

$$\vec{F}_{23} = \int_{-a/2}^{a/2} I(a\hat{z})\times(kz\hat{x})\,dz = 0$$

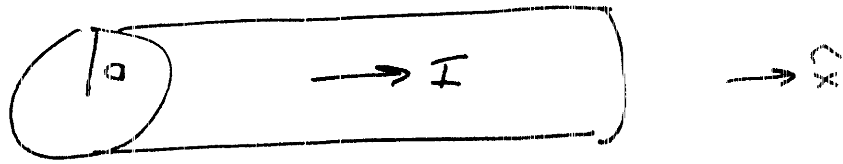
since integration of odd function over even range.

$$= \vec{F}_{41}$$

$$\vec{F}_{34} = I(-a\hat{y})\times\left(\frac{a}{2}k\hat{x}\right) = \frac{Ia^2}{2}k\hat{z}$$

$$\vec{F}_{total} = \sum\vec{F} = \vec{F}_{12} + \vec{F}_{41} = kIa^2\hat{z}$$

5.5



(a) $\vec{K} = \frac{I}{2\pi a}\hat{x} = \frac{I}{\text{circumference}}\hat{x}$

(b) $\vec{J} = \frac{J_0}{s}\hat{x}$

$$I = \int J\,da = \int_0^{2\pi} d\phi \int_0^a s\,ds\left(\frac{J_0}{s}\right)$$

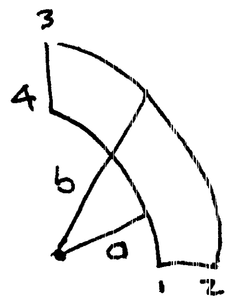
$$= 2\pi J_0 \int_0^a ds = 2\pi J_0 a$$

$$J_0 = \frac{I}{2\pi a}$$

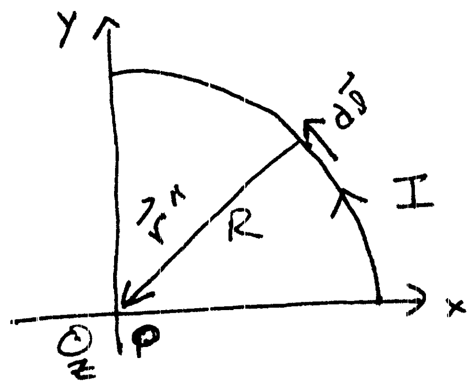
$$\vec{J} = \frac{J_0}{s}\hat{x} = \frac{I}{2\pi a s}\hat{x}$$

5.9

(a)



Magnetic field of quarter circle



Biot-Savart

$$\vec{B}_P = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{r}''}{r''^2}$$

$$d\vec{l}' = R\,d\phi'\,\hat{\phi}' \qquad r'' = R$$

$$|d\vec{l}' \times \hat{r}''| = |d\vec{l}'|\,|\hat{r}''| \sin 90$$

$$= R\,d\phi'$$

$$d\vec{l} \times \hat{r}'' = R\,d\phi'\,\hat{z} \qquad (\text{RHR})$$

$$\vec{B}_p = \frac{\mu_0 I}{4\pi} \frac{R\hat{z}}{R^2} \int_0^{\pi/2} d\phi'$$

$$= \frac{\mu_0 I}{8R} \hat{z}$$

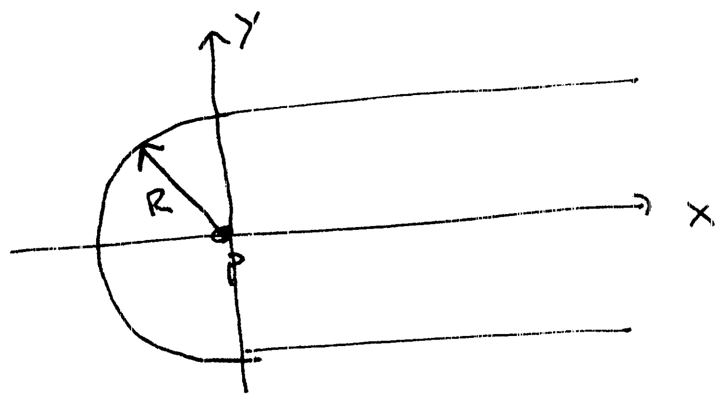
<u>Total Field</u>

$$\vec{B}_p = \underset{0}{\underset{\parallel}{\vec{B}_{12}}} + \vec{B}_{23} + \vec{B}_{34} + \vec{B}_{41}$$

$$\qquad - \frac{\mu_0 I}{8b}\hat{z} + 0 + \frac{\mu_0 I}{8a}\hat{z}$$

$$= \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)$$

(b)



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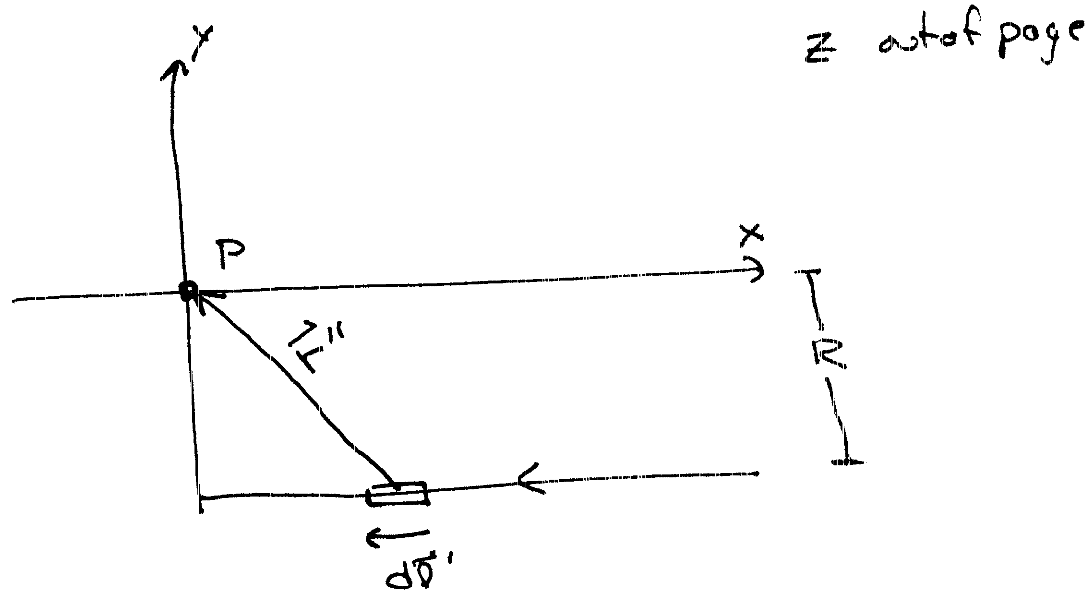
By similar calculation, the field of the half circle is

$$\frac{\mu_0 I}{4R}(-\hat{z})$$

The fields of the two lines are equal and must be calculated by integration.



$\vec{r}_P = (0, 0, 0)$ $\qquad$ $d\vec{l}' = -dx'\hat{x}$ (integrating $0 \to \infty$)

$\vec{r}' = (x', -R, 0)$

$\vec{r}'' = \vec{r}_P - \vec{r}' = (-x', R, 0)$

$r'' = \sqrt{x'^2 + R^2}$

$$\vec{B}_P = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{d\vec{l}' \times \vec{r}''}{r''^3}$$

$$d\vec{l}' \times \vec{r}'' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -dx' & 0 & 0 \\ -x' & R & 0 \end{vmatrix} = -dx' R \hat{z}$$

$$\vec{B}_p = -\frac{\mu_0 I R}{4\pi}\hat{z} \underbrace{\int_0^\infty \frac{dx'}{(x'^2+R^2)^{3/2}}}_{1/R^2}$$

$$= \frac{-\mu_0 I R}{4\pi R^2}\hat{z}$$

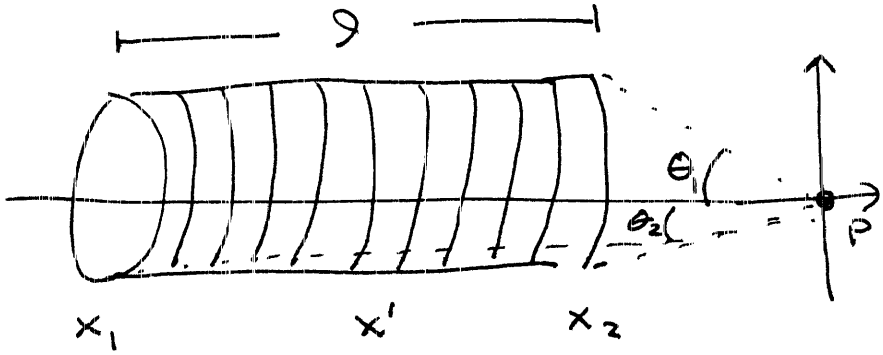
$$= \frac{-\mu_0 I}{4\pi R}\hat{z} \equiv \vec{B}_{wire,\ bottom} = \vec{B}_{wire,\ top}$$

$$\vec{B} = \vec{B}_{circle} + \vec{B}_{wire,\ bottom} + B_{wire,\ top}$$

$$= -\hat{z}\left(\frac{\mu_0 I}{4R} + 2\cdot\frac{\mu_0 I}{4\pi R}\right)$$

$$= -\hat{z}\,\frac{\mu_0 I}{4R}\left(1 + \frac{2}{\pi}\right)$$

5.11



$$K = I\frac{N}{l}$$

Let x' be location of one of the rings

$$dI = K\,dx' = I\frac{N}{l}\,dx'$$

$$\vec{B}_P = \int dB = \int_{x_1}^{x_2} \frac{\mu_0 dI}{2}\,\frac{a^2}{(a^2+x'^2)^{3/2}}\,\hat{x}$$

$$= \mu_0 \frac{N}{l}\,\frac{I}{2}\,a^2 \underbrace{\int_{x_1}^{x_2} \frac{dx'}{(a^2+x'^2)^{3/2}}}\,\hat{x}$$

$$= \frac{\mu_0 N I a^2 \hat{x}}{2l}\left[\frac{x}{a^2\sqrt{x^2+a^2}}\right]_{x_1}^{x_2}$$

$$= \frac{1}{2}\mu_0 \frac{N}{l} I \hat{x}\left(\frac{x_2}{\sqrt{x_2^2+a^2}} - \frac{x_1}{\sqrt{x_1^2+a^2}}\right)$$

$$x_1 = -a\,\cancel{\sin}\cos\theta_2 \qquad x_2 = -a\cos\theta_1$$

$$\frac{x_1}{\sqrt{x_1^2+a^2}} = -\cos\theta_2$$

$$\frac{x_2}{\sqrt{x_2^2+a^2}} = -\cos\theta_1$$

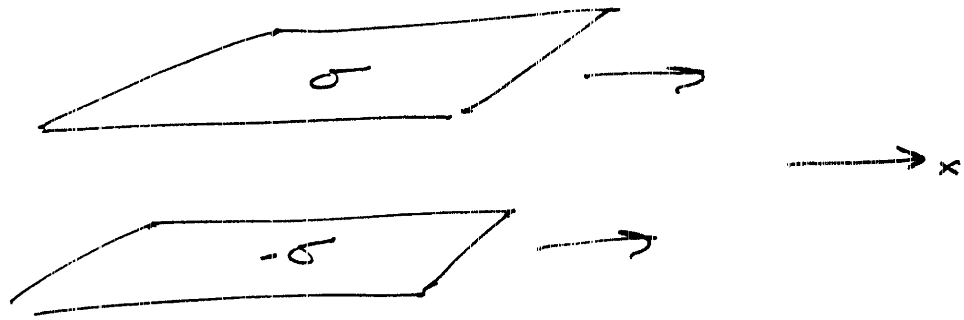
$$\vec{B}_p = \frac{1}{2}\mu_0 \frac{N}{l} I \hat{x} \left(\cos\theta_2 - \cos\theta_1 \right)$$

If solenoid begomes long, $\theta_1 \to \pi$, $\theta_2 \to 0$

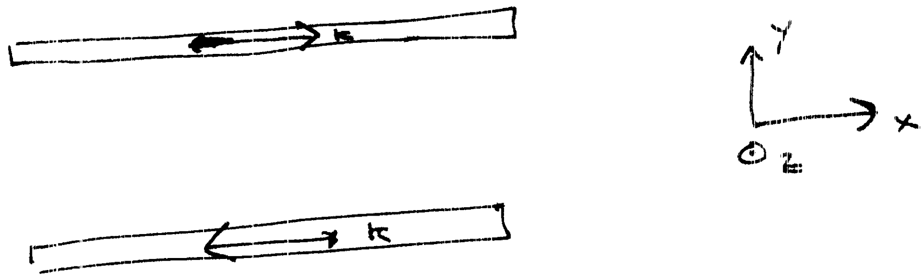
$$(\ ) \to 2$$

$$\vec{B}_p = \mu_0 \frac{N}{l} I \hat{x} \quad \checkmark$$

5.16



$\vec{K}_t = \sigma v \hat{x}$ $\qquad$ $\vec{K}_b = -\sigma v \hat{x}$



The field of one plate is

$$\vec{B} = \begin{cases} \mu_0 \dfrac{K}{2} \hat{z} & \text{above} \\ -\mu_0 \dfrac{K}{2} \hat{z} & \text{below} \end{cases}$$

The fields of the two plates cancel above and below and add in the middle

$$\text{(a)} \qquad \vec{B}_{\text{two plates}} = \begin{cases} 0 & \text{above} \\ -\mu_0 K \hat{z} & \text{between} \\ 0 & \text{below} \end{cases}$$

The magnetic force per unit area on top plate is

$$P_m = \vec{K}_t \times \frac{1}{2}\left(\vec{B}_{above} + \vec{B}_{between}\right)$$

$$= (\sigma v \hat{x}) \times \left(-\frac{1}{2}\mu_0 K \hat{z}\right)$$

$$= \frac{1}{2}\mu_0(\sigma v)^2\, \hat{y} \qquad \text{(upward on plate)}$$

(c) Electric Pressure

$$P_e = \sigma \frac{1}{2}\left(\underbrace{\vec{E}_{above}}_{0} + \underbrace{\vec{E}_{between}}_{\frac{\sigma}{\varepsilon_0}}\right)$$

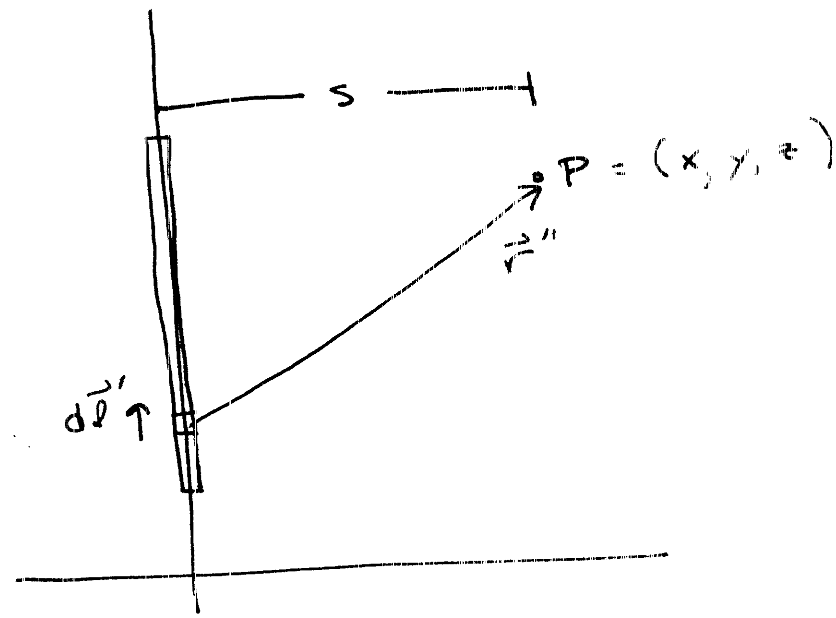
$$= -\frac{1}{2}\frac{\sigma^2}{\varepsilon_0}\,\hat{y}$$

The pressures are equal when

$$\frac{1}{2}\mu_0 \sigma^2 v^2 = \frac{1}{2}\frac{\sigma^2}{\varepsilon_0}$$

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \quad \text{speed of light}$$

5.22



$$\vec{r}'' = s'\hat{s}' + (z-z')\hat{z}$$

$$r'' = \sqrt{s'^2 + (z-z')^2} \quad \hat{z} \qquad\qquad d\vec{l}' = dz'\hat{z}$$

$$\vec{A} = \frac{\mu_0 I \hat{z}}{4\pi} \int_{z_1}^{z_2} \frac{dz'}{\sqrt{s^2 + (z-z')^2}} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}'}{r''}$$

$$= \frac{\mu_0 I \hat{z}}{4\pi} \left[\ln\left(z' - z + \sqrt{s^2 + (z'-z)^2}\right) \right]_{z_1}^{z_2}$$

$$= \frac{\mu_0 I \hat{z}}{4\pi} \ln\left(\frac{z_2 - z + \sqrt{s^2 + (z_2 - z)^2}}{z_1 - z + \sqrt{s^2 + (z_1 - z)^2}} \right)$$

5.23

$$\vec{A} = k\hat{\phi}$$

$$\nabla \times \vec{A} = \underbrace{-\frac{\partial A_\phi}{\partial z}}_{=0}\hat{s} + \underbrace{\frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)}_{=\frac{k}{s}}\hat{z}$$

$$\vec{B} = \frac{k}{s}\hat{z}$$

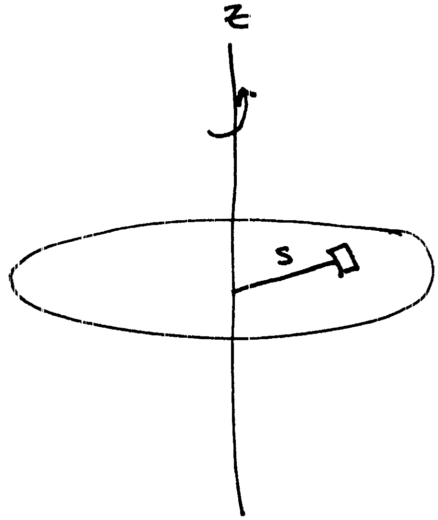
<u>Ampere's Law</u>

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{J} = \frac{1}{\mu_0}\nabla \times \vec{B} = \frac{1}{\mu_0}\left(\underbrace{\frac{1}{s}\frac{\partial B_z}{\partial \phi}}_{=0}\hat{s} - \underbrace{\frac{\partial B_z}{\partial s}}_{=+\frac{k}{s^2}}\hat{\phi}\right)$$

$$\vec{J} = \frac{+k}{\mu_0 s^2}\hat{\phi}$$

5.35



$$\vec{K} = s\omega\sigma\,\hat{\phi}$$

$$d\vec{I} = \vec{K}\,ds$$

$$d\vec{m} = |d\vec{I}|\,A\,\hat{z}$$

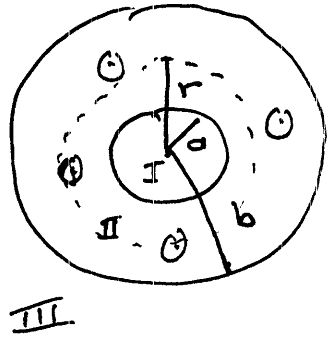
$$A = \pi s^2$$

$$\vec{m} = \int d\vec{m} = \int_0^R (s\omega\sigma)(\pi s^2)\,ds$$

$$= \pi\omega\sigma \int_0^R s^3\,ds$$

$$= \frac{\pi\omega\sigma}{4} R^4$$

E1



Region I $\quad r<a \quad I_{enc}=0 \quad \vec{B}_I=0$

Region II $\quad a<r<b$

$$I_{enc} = \int \vec{J}\cdot\hat{n}\, da$$

$da = (dr)(r\,d\phi) = r\,dr\,d\phi$

$\hat{n} = \hat{z}$

$$= \int_a^r dr \int_0^{2\pi} d\phi\, r\, J$$

$$= \int_a^r dr \int_0^{2\pi} \gamma r^3\, d\phi$$

$$= \frac{\gamma r^4}{4}\Big|_a^r = \frac{\gamma}{4}(r^4 - a^4)$$

Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = 2\pi r B = \mu_0 I_{enc}$$

$$\vec{B} = \frac{\mu_0 I_{enc}}{2\pi r}$$ counterclockwise (from RHR)

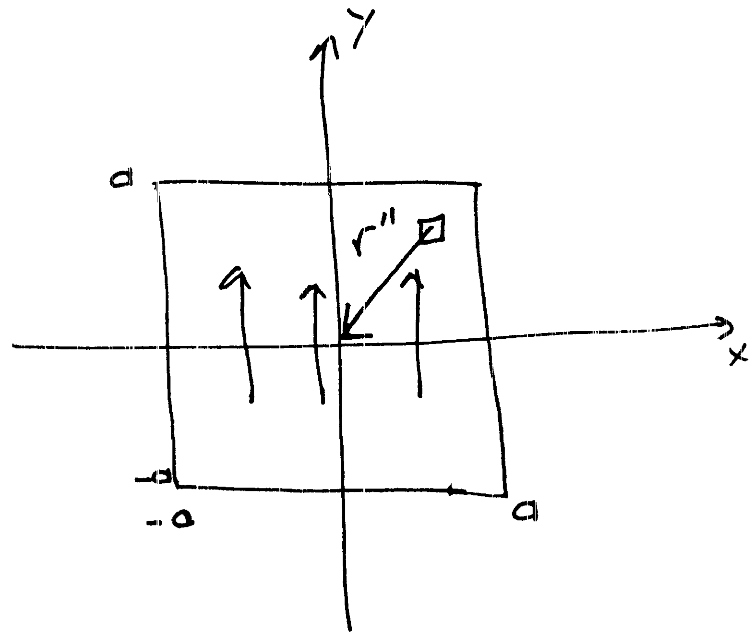
$$\vec{B}_{II} = \frac{\mu_0 \gamma}{8\pi r} \cdot (r^4 - a^4)$$

<u>Region III</u>

$$I_{enc} = \frac{\gamma}{4}(b^4 - a^4)$$

$$\vec{B}_{III} = \frac{\mu_0 I_{enc}}{2\pi r} = \frac{\mu_0 \gamma}{8\pi r}(b^4 - a^4)$$

E2



$$\vec{K} = K_0 \hat{y}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}\, da'}{r''}$$

$\vec{r} = (0,0,0)$ $\quad\quad$ $\vec{r}' = (x', y', 0)$

$$r'' = \sqrt{x'^2 + y'^2}$$

$$da' = dx'\, dy'$$

$$\vec{A} = \frac{\mu_0 K_0}{4\pi} \hat{y} \int_{-a}^{a} dy' \int_{-a}^{a} dx' \frac{1}{\sqrt{x'^2 + y'^2}}$$

$$\vec{A} = \frac{\mu_0 K_0 \hat{y}}{4\pi}\left(4a \ln\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right)\right)$$

$$\vec{A} = \frac{\mu_0 k_0 a}{\pi}\hat{y} \ln\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right)$$

> $assume(a,\ positive);$

> $int\left(int\left(\dfrac{1}{\left(x^2+y^2\right)^{\left(\frac{1}{2}\right)}},\ x=-a..a\right),\ y=-a..a\right);$

$$-4\,a\sim\ln\left(\sqrt{2}-1\right)+4\,a\sim\ln\left(1+\sqrt{2}\right) \tag{1}$$

> $simplify(\%)$

$$-4\,a\sim\left(\ln\left(\sqrt{2}-1\right)-\ln\left(1+\sqrt{2}\right)\right) \tag{2}$$

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