Homework 8

Due Monday 4/19/2010 - at 5:00pm

Reading Assignment - Chapter 8

Griffiths' Problems

7.18

7.24

7.26

7.28

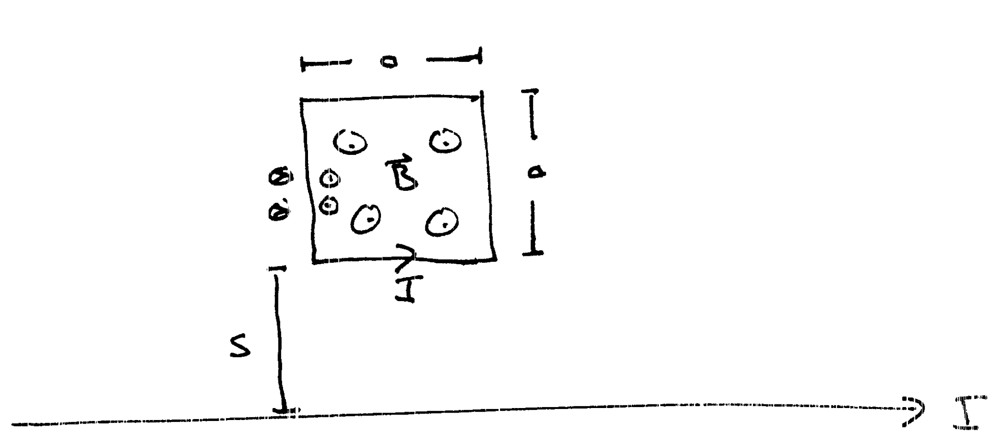
7.33

8.1 Ex. 7.13 only

8.4 (a) only

8.5

7.18



$$I(t) = (1 - \alpha t) I \qquad t < \frac{1}{\alpha}$$

The magnetic field through the loop points out of the page. The flux out of the page decreases, as the current decreases. The induced current produces a flux that opposes this change in flux, the ~~fie~~ field of the current is drawn above. By the RHR, the induced current must flow counterclockwise.

The magnetic field through the loop is

$$B = \frac{\mu_0 I}{2\pi s}$$

The flux is

$$\Phi_m = \cancel{BA = \frac{\mu_0 I a^2}{2\pi s}}$$

$$= a \int_s^{s+a} B \, ds$$

$$\Phi_m = a\int_s^{s+a} \frac{\mu_0 I}{2\pi s}\, ds$$

$$= \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s+a}{s}\right)$$

The emf induced in the loop is, by Faraday

$$\text{emf} = -\frac{d\Phi_m}{dt} = \frac{\mu_0 a}{2\pi} \ln\left(\frac{s+a}{s}\right) \frac{dI}{dt}$$

If the resistance of The loop is R, the current induced is

$$I_{ind} = \frac{\text{emf}}{R} = \frac{\mu_0 a}{2\pi R} \ln\left(\frac{s+a}{s}\right) \frac{dI}{dt}$$

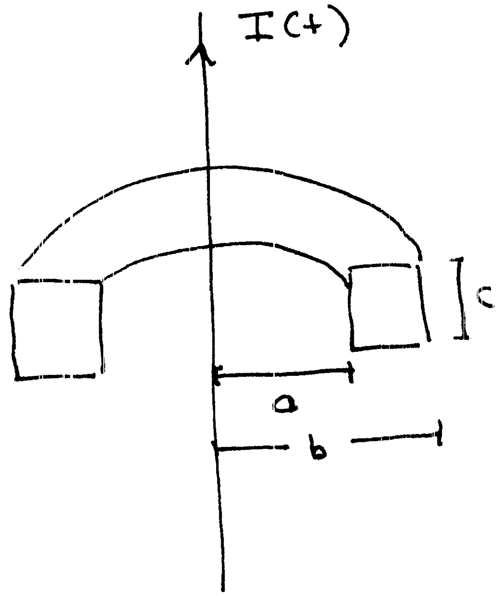
The total charge that flow past some point in the loop over time is the integral of the current

$$|Q| = \left|\int I_{ind}\, dt\right| = \left|\frac{\mu_0 a}{2\pi R} \ln\left(\frac{s+a}{s}\right) \frac{dI}{dt}\, dt\right|$$

$$= \frac{\mu_0 a}{2\pi R} \ln\left(\frac{s+a}{s}\right) \left|\int_{I_0}^{0} dI\right|$$

$$= \frac{\mu_0 a I_0}{2\pi R} \ln\left(\frac{s+a}{s}\right)$$

7.24



$N = 1000$

$a = 1\text{cm}$

$b = 2\text{cm}$

$c = 1\text{cm}$

$I = I_0 \cos \omega t$

$I_0 = \frac{1}{2} A$

$f = 60\ Hz$

$\omega = 2\pi f$

$R = 500\ \Omega$

Magnetic field of the wire

$$B_w = \frac{\mu_0 I}{2\pi s}$$

The magnetic flux through the solenoid

$$\Phi_m = Nc \int_a^b B\, ds$$

$$= \frac{\mu_0 N I c}{2\pi} \ln\left(\frac{b}{a}\right)$$

The emf induced in the toroid is by Faraday

$$emf = -\frac{d\Phi_m}{dt} = -\frac{\mu_0 N c}{2\pi} \ln\left(\frac{b}{a}\right) \frac{dI}{dt}$$

$$= \frac{I_0 \mu_0 N C}{2\pi} \omega \ln\left(\frac{b}{a}\right) \sin \omega t$$

$$= I_0 \mu_0 N c f \ln\left(\frac{b}{a}\right) \sin \omega t$$

Putting in the hated numbers,

$$emf_{max} = (0.5A)\left(4\pi\times10^{-7}\frac{Tm}{A}\right)(1000)(0.01m)(60\,s^{-1})\ln(2)$$

$$= 2.6\times10^{-4}V$$

So the current induced is

$$I_{max} = \frac{emf_{max}}{R} = 5.2\times10^{-7}A$$

$$I(t) = I_{max}\sin\omega t$$

(b) The back current can be calculated from the self-inductance given in example 7.11

$$L = \frac{\mu_0 N^2 c}{2\pi}\ln\left(b/a\right)$$

$$= \frac{\left(4\pi\times10^{-7}\frac{Tm}{A}\right)(1000)^2(0.01)}{2\pi}\ln(2)$$

$$= 1.386\times10^{-3}H$$

Back emf

$$\mathcal{E}_{back} = -L\frac{dI_{ind}}{dt} = \underbrace{-L\,I_{max}}_{I_{back,max}}\,\omega\cos\omega t$$

$$emf_{back, max} = -L I_{max} \omega = -2\pi f L I_{max}$$

$$= -2\pi (60 s^{-1})(5.2\times 10^{-7} A)(1.386\times 10^{-3} H)$$

$$= 2.7\times 10^{-7} V$$

The back current

$$I_{back, max} = \frac{emf_{back, max}}{R} = 5.43\times 10^{-10} A$$

Ratio of back emf to direct emf

$$= \frac{2.7\times 10^{-7} V}{2.6\times 10^{-4} V} \sim \frac{1}{1000}$$

7.26

(a) $L = \mu_0 n^2 A l$

$$U = \frac{1}{2} L I^2 = \frac{1}{2}\mu_0 n^2 A l I^2 = \frac{1}{2}\mu_0 n^2 \pi R^2 l I^2$$

(b) $W = \frac{1}{2}\oint \vec{A}\cdot\vec{I}\, d\vec{l}$

$$\vec{A} = \frac{\mu_0 n I}{2} R\,\hat{\phi}$$

The integral is carried out around the circumference of one ring of thickness dz and gives the energy in one ring

$$d\vec{I} = K\, dz\,\hat{\phi} = nI\, dz\,\hat{\phi}$$

$$dW = \frac{1}{2}\oint(\vec{A}\cdot d\vec{I})\, d\vec{l}$$

$$W = \int_0^l dW = \frac{1}{2}\underbrace{\int_0^l dz}_{l}\underbrace{\oint_c \vec{A}\cdot(nI\hat{\phi})\, d\vec{l}}_{2\pi R A n I}$$

$$= \frac{1}{2} l\cdot 2\pi R\cdot\frac{\mu_0 n I R}{2} nI$$

$$= \frac{1}{2} n^2 I^2 \mu_0 \pi R^2 l$$

(c)

$$W = \frac{1}{2\mu_0} \int B^2 \, d\tau = \frac{1}{2\mu_0} B^2 \cdot \pi R^2 l$$

$$= \frac{1}{2\mu_0} \cdot (\mu_0 n I)^2 \cdot \pi R^2 l = \frac{1}{2} \mu_0 n^2 I^2 \pi R^2 l$$

(d)

$$W = \frac{1}{2\mu_0} \left[\int_V B^2 \, d\tau - \oint_S (\vec{A} \times \vec{B}) \cdot d\vec{a} \right.$$

where the volume V is $a < s < b$

where $a < R$ and $b > R$.

The first term is as before

$$\frac{1}{2\mu_0} \int B^2 \, d\tau = \frac{1}{2\mu_0} (\mu_0 n I)^2 \cdot \pi (R^2 - a^2) l$$

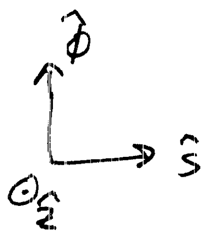
where $\pi (R^2 - a^2) l$ is the volume of the region

where the field is non-zero.

The second term

$$\vec{A} \times \vec{B} = \frac{\mu_0 n I}{2} s \, \hat{\phi} \times \mu_0 n I \, \hat{z} \qquad s < R$$

$$= 0 \qquad s > R.$$



$$\hat{\phi} \times \hat{z} = \hat{s}$$

$$\vec{A} \times \vec{B} = \frac{1}{2} \mu_0^2 n^2 I^2 s \; \hat{s} \qquad s < R$$

$\int_S (\vec{A} \times \vec{B}) \cdot d\vec{a}$ must be taken over the inner and outer surface. The integral over the outer surface is zero. For the inner surface, $d\vec{a} = -\hat{s}\, da$

$$\frac{1}{2\mu_0} \int_S \vec{A} \times \vec{B} \cdot (-\hat{s}\, da) =$$

$$= -\frac{1}{2\mu_0} \cdot \frac{1}{2} \mu_0^2 n^2 I^2 a \underbrace{\int_S da}_{2\pi a l}$$

$$= -\frac{1}{2\mu_0} \cdot \frac{1}{2} \mu_0^2 n^2 I^2 a \cdot 2\pi a l$$

$$= -\frac{1}{2} \mu_0 n^2 I^2 \pi a^2 l$$

$$W = \frac{1}{2\mu_0} \int_V B^2 d\tau - \frac{1}{2\mu_0} \int_S \vec{A} \times \vec{B} \cdot d\vec{a}$$

$$= \frac{1}{2} \mu_0 n^2 I^2 \pi R^2 l$$

7.28 There is no obvious correct surface through which the flux should be calculated. As such, it is unclear how to compute L from Φ_m / I.

We can however unambiguously compute the energy U and find L from $U = \frac{1}{2} L I^2$.

The magnetic field in the wire with $J = I/\pi R^2$

<u>Ampere's Law</u>

$$\oint \vec{B} \cdot d\vec{l} = 2\pi s B = \mu_0 I_{enc} = \mu_0 \pi s^2 J = \mu_0 \frac{s^2}{R^2}$$

$$B = \frac{\mu_0 I s}{2\pi R^2} \quad \text{as before}$$

The energy in a length l of the wire is

$$U = \frac{1}{2\mu_0} \int_V B^2 \, d\tau = \frac{l}{2\mu_0} \int B^2 \, da$$

$$da = ds \, s \, d\phi$$

$$= \frac{1}{2\mu_0} \int \left(\frac{\mu_0 I s}{2\pi R^2} \right)^2 \cdot s \, ds \, d\phi$$

$$= \frac{1}{2\mu_0} \left(\frac{\mu_0 I}{2\pi R^2} \right)^2 \int_0^{2\pi} d\phi \int_0^R s^3 \, ds$$

$$U = \frac{l}{2\mu_0}\left(\frac{\mu_0 I}{2\pi R^2}\right)^2 \cdot 2\pi \cdot \frac{R^4}{4}$$

$$= \frac{1}{2}\left(\frac{\mu_0 l}{8\pi}\right) I^2 = \frac{1}{2} L I^2$$

$$L = \frac{\mu_0 l}{8\pi}$$

7.33

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$= \frac{\epsilon_0 \mu_0 I_0 \omega}{2\pi} \ln\left(\frac{a}{s}\right) \hat{z} \frac{\partial}{\partial t} \sin\omega t$$

$$= + \frac{\epsilon_0 \mu_0 I_0 \omega^2}{2\pi} \ln\left(\frac{a}{s}\right) \cos\omega t \, \hat{z}$$

(b) Total displacement current

$$I_d = \int J_d \cdot d\vec{a} \qquad da = s\, ds\, d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^a s\, J_d\, ds$$

$$= 2\pi \cdot \left(\frac{+\epsilon_0 \mu_0 I_0 \omega^2}{2\pi} \cos\omega t \right) \underbrace{\int_0^a s \ln\left(\frac{a}{s}\right) ds}_{+a^2/4}$$

$$= \epsilon_0 \mu_0 \omega^2 I_0 \cos\omega t \cdot \left(+ \frac{a^2}{4} \right)$$

$$= \frac{1}{4} \epsilon_0 \mu_0 a^2 \omega^2 I_0 \cos\omega t$$

(c) $I = I_0 \cos \omega t$

$$\frac{I_d}{I} = \frac{1}{4}\varepsilon_0 \mu_0 a^2 \omega^2$$

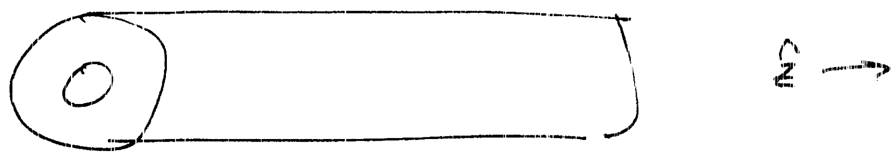
If $\dfrac{I_d}{I} = 0.01 = \dfrac{1}{4}\mu_0 \varepsilon_0 a^2 (2\pi f)^2$

$$= \mu_0 \varepsilon_0 a^2 \pi^2 f^2 \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$f = \left(\frac{0.01}{\mu_0 \varepsilon_0 a^2 \pi^2}\right)^{1/2} = \frac{1}{10}\frac{c}{a\pi}.$$

$$= \frac{3\times 10^8 \, m/s}{10 \cdot (0.002 m) \cdot \pi} = 1.5 \times 10^9 \, Hz$$

8.1



Magnetic field — $\dfrac{\mu_0 I}{2\pi s}\,\hat{\phi}$ Ex 7.13

Electric field — If a potential difference ΔV is established between the inner and outer conductor, a charge density λ is established on inner conductor. The electric field between the conductors is then

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\,\hat{s}$$

~~The po.~~ The potential difference between the conductors is then

$$\Delta V = -\int_a^b \vec{E}\cdot d\vec{l} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

The Poynting vector is

$$|\vec{S}| = \frac{1}{\mu_0}\,|\vec{E}\times\vec{B}| = \frac{EB}{\mu_0}$$

since $\vec{E}\perp B$

The energy flow down the channel per unit time, the power is

$$P = \int S\, da = 2\pi \int_a^b s\, S\, ds$$

$$S = \frac{1}{\mu_0} E B = \left(\frac{1}{\mu_0}\right)\left(\frac{\lambda}{2\pi\epsilon_0 s}\right)\left(\frac{\mu_0 I}{2\pi s}\right)$$

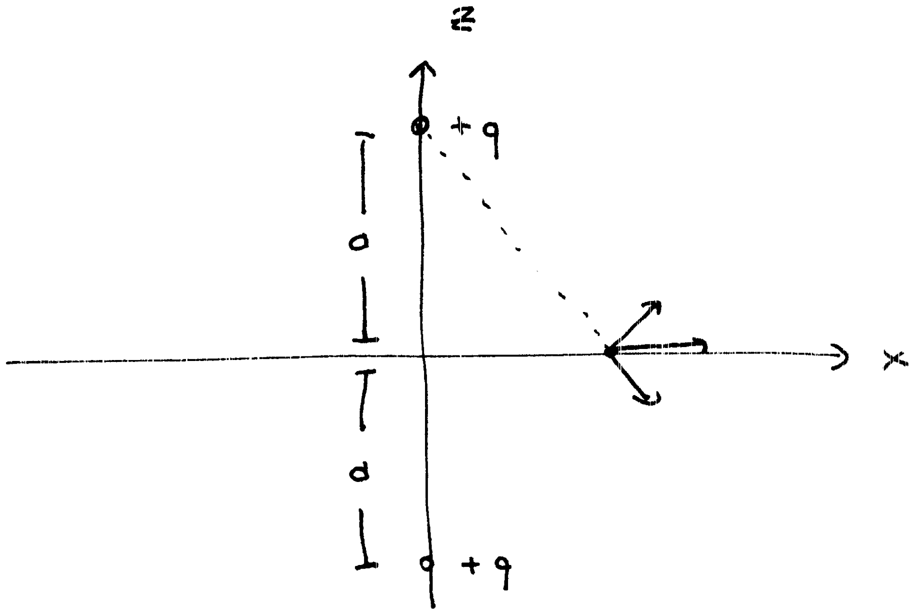
$$= \frac{\lambda I}{4\pi^2 \epsilon_0 s^2}$$

$$P = 2\pi \int_a^b \frac{\lambda I}{4\pi^2 \epsilon_0 s^2} \cdot s\, ds$$

$$= \frac{\lambda I}{2\pi\epsilon_0} \int_a^b \frac{ds}{s} = \frac{\lambda I}{2\pi\epsilon_0} \ln\left(b/a\right)$$

$$= I \Delta V \quad \checkmark$$

8.4 (a)



The electric field of the two charges on the plane is

$$\vec{E} = \frac{2kq}{s^2+a^2} \cdot \frac{s}{\sqrt{s^2+a^2}} \hat{s}$$

$$= \frac{2kqs}{(s^2+a^2)^{3/2}} \hat{s}$$

$$\vec{F} = \int \overleftrightarrow{T} \cdot d\vec{a}$$

$$d\vec{a} = \hat{z}\, da$$

$$\overleftrightarrow{T} \cdot d\vec{a} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ da \end{pmatrix}$$

$$\vec{F} = \begin{pmatrix} T_{xz}\, da \\ T_{yz}\, da \\ T_{zz}\, da \end{pmatrix}$$

The force must be in the positive z direction,

so $\int T_{xz}\, da = \int T_{yz}\, da = 0$

$$T_{zz} = \epsilon_0\left(E_z^2 - \frac{1}{2} E^2\right) + \frac{1}{\mu_0}\left(B_z^2 - 1/2\, B^2\right)$$

$$= \epsilon_0\left(E_z^2 - \frac{1}{2}\vec{E}\cdot\vec{E}\right)$$

$E_z = 0$, $\vec{E}\cdot\vec{E} = \dfrac{4k^2q^2s^2}{(s^2+a^2)^3} = \dfrac{q^2s^2}{4\pi^2\epsilon_0^2(s^2+a^2)^3}$

$$T_{zz} = -\frac{1}{2}\epsilon_0\, \vec{E}\cdot\vec{E} = -\frac{q^2s^2}{8\pi^2\epsilon_0(s^2+a^2)^3}$$

To calculate the force on the upper charge we need the <u>outward</u> normal from the top plane

$$\hat{n} = -\hat{z}$$

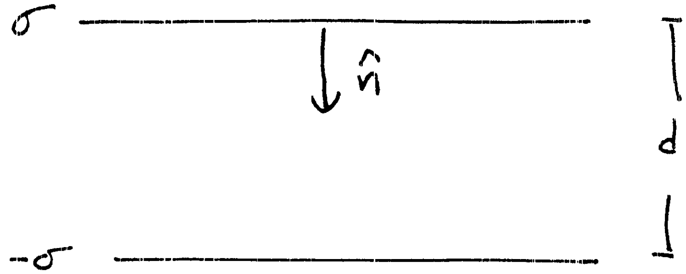
$$\vec{F} = (-\hat{z}) \int T_{zz} \, da$$

$$= \frac{q^2 \hat{z}}{8\pi^2 \epsilon_0} \int_{\text{plane}} \frac{s^2}{(s^2+a^2)^3} \, ds \, s \, d\phi$$

$$= \frac{2\pi q^2 \hat{z}}{8\pi^2 \epsilon_0} \underbrace{\int_0^\infty \frac{s^3}{(s^2+a^2)^3}}_{\frac{1}{4a^2}}$$

$$\vec{F} = \frac{q^2 \hat{z}}{16\pi a^2 \epsilon_0} = \frac{q^2}{4\pi\epsilon_0 (2a)^2} \hat{z} \quad \checkmark$$

8.5



$$\vec{E} = -\frac{\sigma}{\epsilon_0}\,\hat{z}$$

(a) $$T_{ij} = \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$E_{zz} = \frac{-\sigma}{\epsilon_0} \qquad \text{All other } E_{ij},\ B_{ij} = 0$$

$$T_{ij} = 0 \quad \text{if} \quad i \neq j$$

$$T_{xx} = \epsilon_0\left(-\frac{1}{2}E^2\right) = -\frac{\sigma^2}{2\epsilon_0} = T_{yy}$$

$$T_{zz} = \epsilon_0\left(E_z^2 - \frac{1}{2}E^2\right) = \frac{\sigma}{2\epsilon_0}$$

$$\overleftrightarrow{T} = \frac{\sigma}{2\epsilon_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) The force on the top plate

$$\vec{F} = \int \overleftrightarrow{T} \cdot d\vec{a} \qquad d\vec{a} = -\hat{z}\,dx\,dy$$

Force per unit area, $\vec{f} = \dfrac{\vec{F}}{da} = \overleftrightarrow{T} \cdot (-\hat{z})$

$$= -\hat{z}\left(\frac{\sigma^2}{2\epsilon_0}\right)$$

$$\vec{f} = \frac{\sigma^2}{2\epsilon_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \frac{\sigma^2}{2\epsilon_0}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

(c) The momentum per unit area per unit time crossing x-y plane is $-T_{zz} = -\dfrac{\sigma^2}{2\epsilon_0}$

(d) The recoil force ~~is~~ is $\dfrac{dp}{dt} = -T_{zz} = \dfrac{-\sigma^2}{2\epsilon_0}\hat{z}$