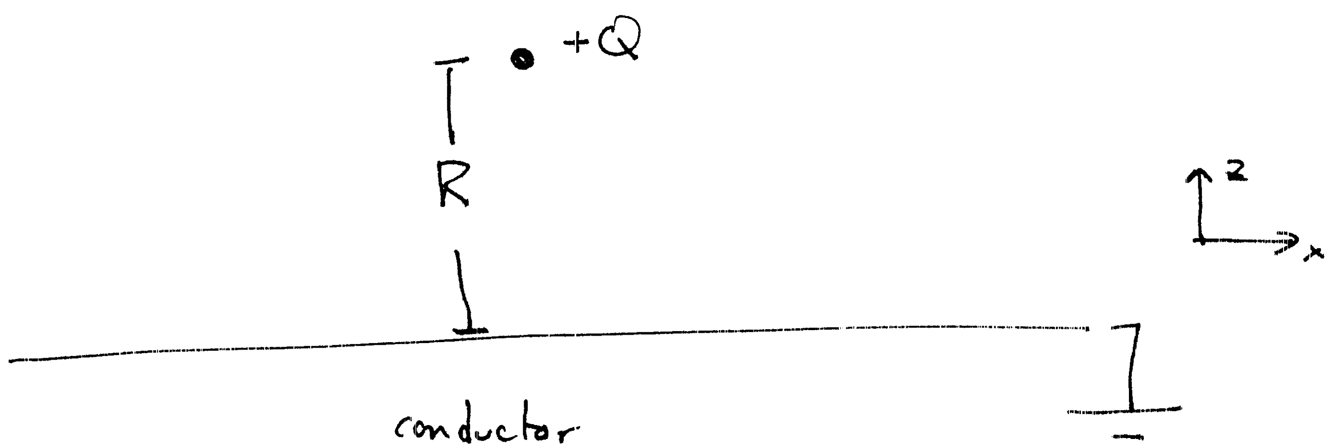
Method of Images

The method of images solves Poisson's eqn in a region R by satisfying the boundary conditions using <u>fictitious</u> charges placed outside of R.

<u>Ex</u> Consider a point charge above a grounded conducting plane.

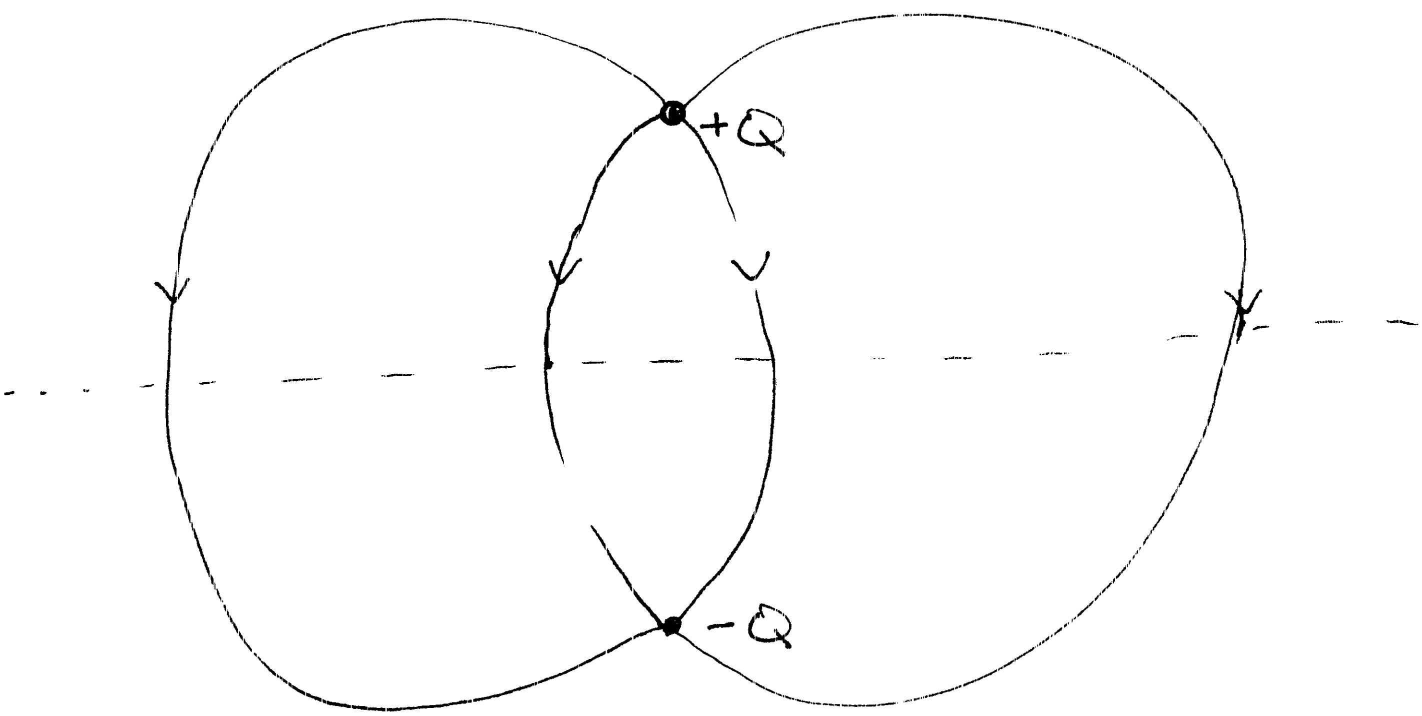


The boundary conditions on the region $z > 0$ are

$$V(x, y, 0) = 0 \qquad V(x, y, z) \to 0 \text{ as } r \to \infty.$$

The uniqueness thm guarantees that if we can find any solution to Poisson's equation that satisfies the boundary conditions, that it is the unique solution.

Consider a dipole field



The potential of the two charges is

$$V(x,y,z) = \frac{kQ}{r_+} + \frac{-kQ}{r_-}$$

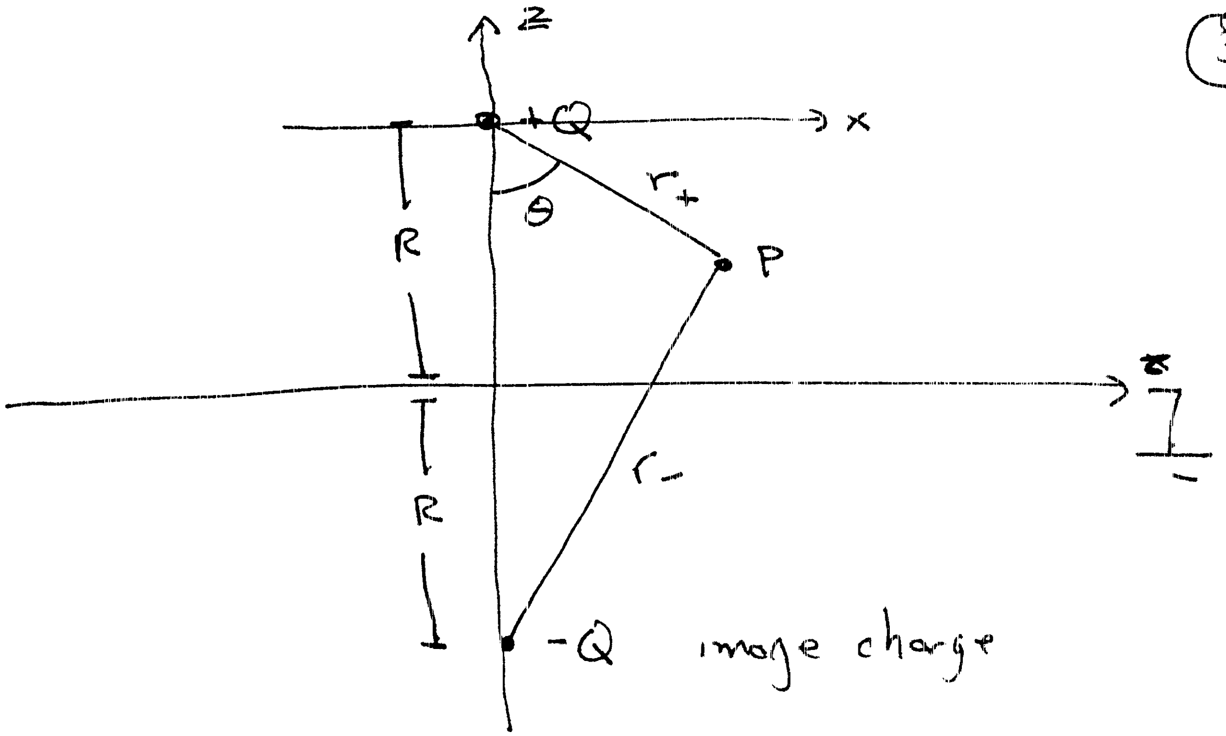
For points on the plane equidistant to both charges, the potential is zero. This is exactly what we need to solve the boundary conditions for the single point charge above the conducting plane.

Therefore, by uniqueness the potential of a point charge q above a grounded conducting plane is

$$V = \frac{KQ}{r_+} + \frac{-KQ}{r_-}$$

for $z > 0$.

The $-Q$ charge is called an <u>image charge</u>. It is not real and it must be placed <u>outside</u> the region where we need the potential.



$$V(r_+, \theta) = \frac{kQ}{r_+} - \frac{kQ}{r_-}$$

$$r_-^2 = r_+^2 + 4R^2 - 4r_+ R \cos\theta$$

$$V(r_+, \theta) = \frac{kQ}{r_+} - \frac{kQ}{\sqrt{r_+^2 + 4R^2 - 4r_+ R\cos\theta}}$$

$$= \frac{kQ}{r_+}\left(1 - \frac{1}{\sqrt{1 + \left(\frac{2R}{r_+}\right)^2 - \frac{4R}{r_+}\cos\theta}}\right)$$

Not the prettiest thing in the world, but consider the alternative of solving this as an expansion of Legendre polynomials in spherical coordinates.

Since the potential is the same for the two systems, the fields are the same and all mechanical properties carry over (almost).

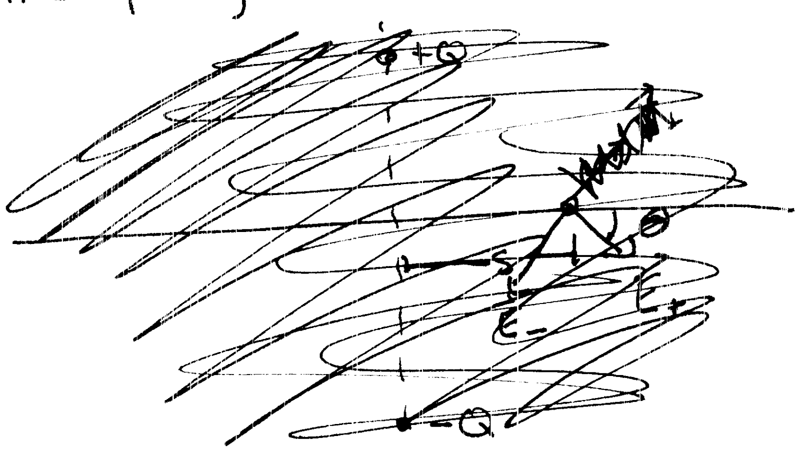
The force the ~~induced~~ surface charge on the plane exerts on +Q charge is the force the −Q charge exerts on the +Q charge.

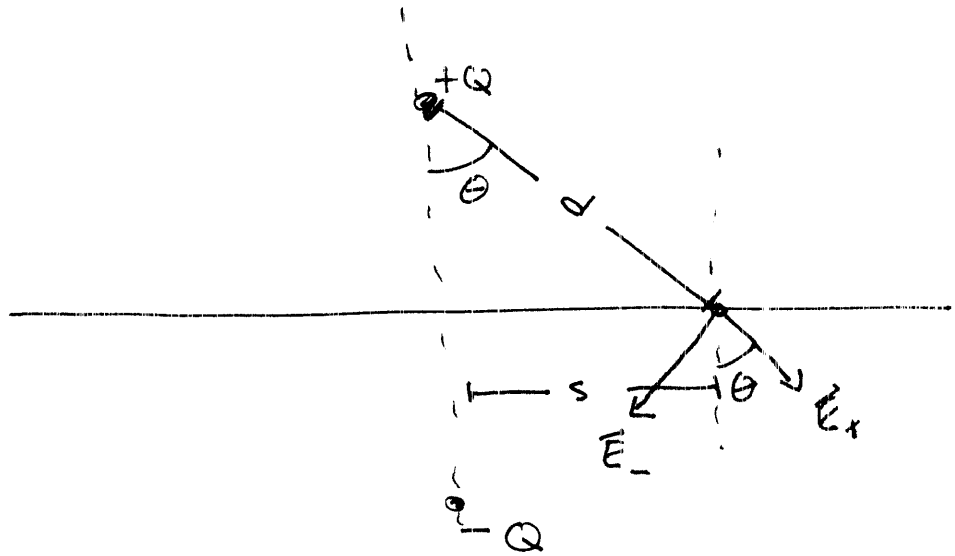
$$\vec{F}_+ = \frac{K(-Q)(+Q)}{(2R)^2}\,\hat{z} = \frac{-KQ^2}{4R^2}\,\hat{z}$$

The field is the sum of the two point charge fields,

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

At the plane, the x, y components are zero.





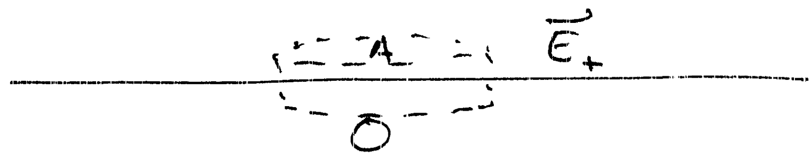
$$\vec{E}_{plane} = \vec{E}_+ + \vec{E}_- = -2|E_+|\cos\theta\,\hat{z}$$

$$E_+ = \frac{kQ}{d^2} = \frac{kQ}{s^2+R^2}$$

$$\cos\theta = \frac{R}{d} = \frac{R}{\sqrt{s^2+R^2}}$$

$$\vec{E}_{plane} = -\frac{2kQR}{(s^2+R^2)^{3/2}}\hat{z} = -\frac{QR\hat{z}}{2\pi\varepsilon_0(s^2+R^2)^{3/2}}$$

The charge density on the conductor is found using a Gaussian pillbox,



$$\vec{E}_+ \cdot \hat{n} A - 0 = \frac{\sigma A}{\varepsilon_0}$$

$$\frac{-QR}{2\pi\varepsilon_0 (s^2+R^2)^{3/2}} = \frac{\sigma}{\varepsilon_0}$$

$$\sigma = \frac{-QR}{2\pi (s^2+R^2)^{3/2}}$$

The total charge on the plane is

$$Q_{plane} = \int_0^{2\pi} d\phi \int_0^{\infty} s\,ds\,\sigma \qquad da = s\,d\phi\,ds$$

$$= 2\pi \int_0^{\infty} \sigma s\,ds$$

$$Q_{plane} = -QR \int_0^\infty \frac{s}{(s^2+R^2)^{3/2}}\, ds$$

$$u = s^2 + R^2 \qquad du = 2s\,ds$$

$$Q_{plane} = -\frac{QR}{2}\int_{R^2}^{\infty} \frac{du}{u^{3/2}} = -\frac{QR}{2}\left(\frac{u^{-1/2}}{-1/2}\right)\Bigg|_{R^2}^{\infty}$$

$$= -\frac{RQ}{\sqrt{R^2}} = -Q \quad \checkmark$$

Since the plane extends to ∞, no field lines can escape to ∞, since the plane is grounded. All lines starting at $+Q$ end on the plane $\Rightarrow$ The plane must have charge $-Q$.

The energy of the $+Q$ / plane system is

$$U = \frac{1}{2}\left(\frac{KQ^2}{2R}\right)$$

which is ½ the energy of the $\pm Q$ dipole.

The energy is 1/2 because 1/2 of the fields of the charges do not exist.

Note, we could build up the potential of a charged conducting plane with average surface charge density σ_0 by using an image charge to bring the plane all to zero potential and then ~~using~~ adding a uniform plane of charge with constant charge density σ_0.