Integrals

To build up the electric or magnetic field of a complicated system, we will integrate over the source charges or currents. This involves integrating over curves, surfaces, and volumes.

Line Integrals

$$\int_{\text{curve}} f \, dl$$

Surface Integrals

$$\int_{\text{surface}} f \, da$$

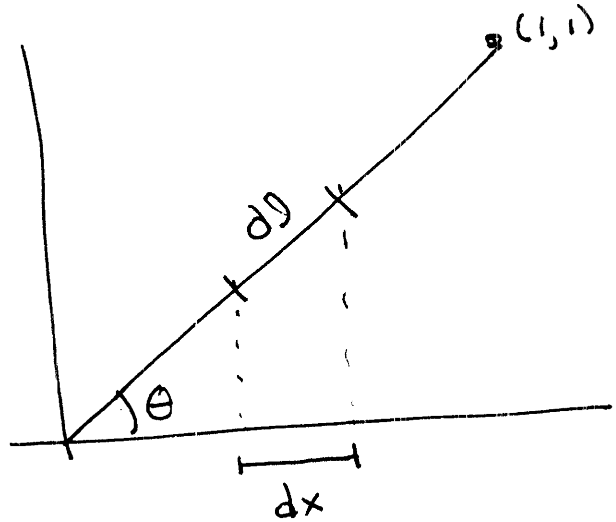
Volume Integrals

$$\int_{\text{volume}} f \, d\tau$$

To evaluate any of these, imagine dividing the object into small pieces.

<u>Ex</u> Compute $\int d\ell$ along the line from $(0,0)$ to $(1,1)$

Note, this is obviously just the length of the line $\sqrt{2}$.

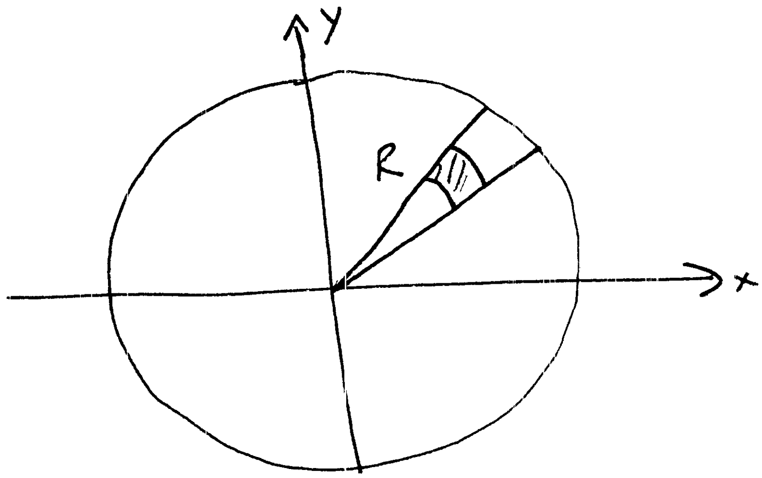


<u>Sln</u> We don't integrate over the line directly, but over one of the coordinate lines, so we have to express $d\ell$ in terms of dx. Evidently, from the figure

$$d\ell = dx/\cos\theta = \sqrt{2}\,dx$$

$$\int_{\text{line}} d\ell = \int_0^1 \sqrt{2}\,dx = \sqrt{2} \quad ✓$$

Ex Compute area of a circle of radius R



$$\text{Area} = \int_{\text{circle}} da$$

We can evaluate this in any coordinate system. It is somewhat complicated in cartesian where $da = dx\,dy$. A small area element is drawn above. Try cylindrical.

$$da = (dr)(r\,d\theta)$$

$$\text{Area} = \int_{\text{circle}} da = \int_0^R dr \int_0^{2\pi} r\,d\theta$$

$$= \int_0^R 2\pi r\,dr = \pi R^2$$

Integral Theorems

We will convert UPII into this class using a number of integral theorems.

Gradient

The total derivative of a function can be written

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

The integral of df is

$$\int_{\vec{r}_a}^{\vec{r}_b} df = f(\vec{r}_b) - f(\vec{r}_a)$$

In Cartesian coordinates, the path element is $d\vec{l} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$, the infinitesimal displacement.

therefore

$$df = \nabla f \cdot d\vec{l}$$

The line integral, where $d\vec{l}$ is an element of the line of the gradient is

$$I \equiv \int_{\substack{\vec{r}_a \to \vec{r}_b \\ \text{path}}} \nabla f \cdot d\vec{l} = \int_{\vec{r}_a}^{\vec{r}_b} df = f(\vec{r}_b) - f(\vec{r}_a)$$

- $\Rightarrow$ I is independent of the path. It depends only on the end points.

  $\Rightarrow$ If the path is closed $\vec{r}_a = \vec{r}_b$, $I = 0$.

Divergence Thm

$$\int_{\text{volume}} (\nabla \cdot \vec{A})\, d\tau = \oint_{\text{surface}} \vec{A} \cdot d\vec{a}$$

- surface = closed surface surrounding volume
- $d\vec{a} = \hat{n}\, da$ where $\hat{n}$ is outward surface normal; a vector $\perp$ to the surface.

<u>Flux</u> The flux or flow of the field $\vec{A}$ out of a closed surface S is defined a

$$\Phi = \int_S \vec{A} \cdot d\vec{a}$$

So the integral of the divergence of the field over a volume equals the flow out of the volume.

<u>Stoke's Thm</u> The integral of the field $\vec{A}$ around a closed curve C is equal to the total curl of the field over the surface S bounded by C.

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot d\vec{a}$$

closed

$\Rightarrow$ Chose $\hat{n}$ for S by curling fingers in direction of C, thumb points in $\hat{n}$ direction.

There are many other integral formulas

$$\int_{\text{volume}} (\nabla f)\, d\tau = \oint_S f\, d\vec{a}$$

$$\int_S \nabla f \times d\vec{a} = -\oint_C f\, d\vec{l}$$

$$\int_V (\nabla \times \vec{A})\, d\tau = -\oint_S \vec{A} \times d\vec{a}$$

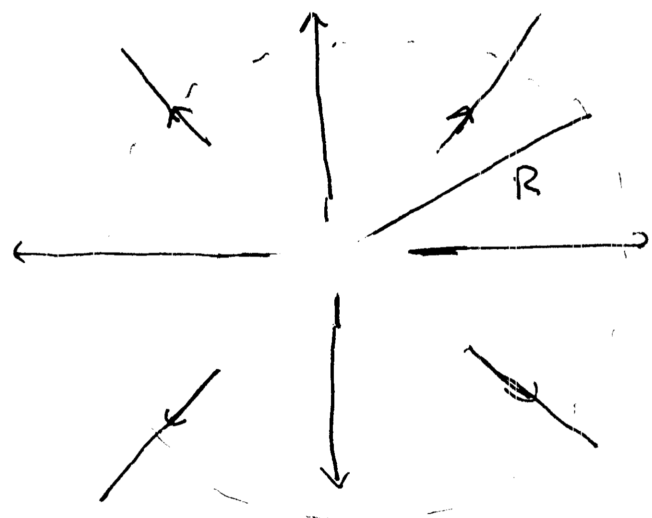
$$\int_S (d\vec{a} \times \nabla) \times \vec{A} = -\oint_C \vec{A} \times d\vec{l} \qquad (\text{Stokes } 2)$$

$$\int_V (f \nabla^2 g + \nabla f \cdot \nabla g)\, d\tau = \oint_S (f \nabla g) \cdot d\vec{a}$$

(Green 1)

$$\int_V (f \nabla^2 g - g \nabla^2 f)\, d\tau = \oint_S (f \nabla g - g \nabla f) \cdot d\vec{a}$$

<u>Ex</u> Consider the field $\vec{A} = \gamma \vec{r}$. The field points outward from the origin. It grows stronger as you move away from the origin.



The flux out of a sphere of radius R is

$$\Phi = \oint_{\text{surface}} \vec{A} \cdot d\vec{a}$$

$$d\vec{a} = \hat{r} \; r\sin\theta \, d\phi \; r\, d\theta = \hat{r} R^2 \sin\theta \, d\phi \, d\theta$$

$$\Phi = \oint (\gamma \vec{r}) \cdot \hat{r} \; R^2 d\phi \, d\theta \sin\theta$$

$$\vec{r} = r\hat{r} \qquad \vec{r} \cdot \hat{r} = r = R$$

$$\Phi = \gamma R^3 \int_0^{2\pi} d\phi \underbrace{\int_0^{\pi} d\theta \sin\theta}_{-\cos\theta \Big|_0^{\pi}}$$

$$2$$

$$\Phi = 2\gamma R^3 \int_0^{2\pi} d\phi$$

$$= 4\pi R^3 \gamma \quad = (4\pi R^2)(\gamma R)$$

$$= \text{surface area} \cdot A$$

~~$\nabla \cdot \vec{A} =$~~ Now check divergence

$$\vec{A} = \underbrace{\gamma r}_{A_r} \hat{r} + 0\hat{\theta} + 0\hat{\phi}$$

Front Cover $\quad \nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r)$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}(\gamma r^3)$$

$$= \frac{3\gamma r^2}{r^2} = 3\gamma$$

$$\int_{\text{volume}} \nabla \cdot \vec{A}\, d\tau = \int_{\text{volme}} 3\gamma\, d\tau$$

$$= 3\gamma \int_{\text{volume}} d\tau = 3\gamma \cdot \frac{4}{3}\pi R^3$$

$$= 4\pi R^3 \gamma \quad \checkmark$$

<u>Ex</u> Investigate Stoke's Thm Pick a field that is purely rotational

$$\vec{A} = \gamma \hat{\phi}$$

and apply Stoke's Thm for a circular surface of radius R. $d\vec{l} = R\,d\phi\,\hat{\phi}$

$$\oint_c \vec{A} \cdot d\vec{l} = \oint_c \vec{A} \cdot (R\hat{\phi}\,d\phi)$$

$$= \oint_c \gamma\hat{\phi} \cdot (R\hat{\phi}\,d\phi) = R\gamma \oint_c d\phi$$

$$= 2\pi R \gamma$$

Compute Curl (cylindrical)

$$A_s = 0, \quad A_\phi = \gamma, \quad A_z = 0$$

$$\nabla \times \vec{A} = -\frac{\partial A_\phi}{\partial z}\,\hat{s} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s A_\phi)\right]\hat{z}$$

$$= \frac{1}{s}\,\frac{\partial}{\partial s}(s\gamma)\,\hat{z} = \frac{\gamma}{s}\,\hat{z}$$

Compute

$$d\vec{a} = s\,d\phi\,ds\,\hat{z}$$

$$\int_{\text{circle}} \nabla \times \vec{A} \cdot d\vec{a} = \int_0^{2\pi} d\phi \int_0^R s\,ds\left(\frac{\gamma}{s}\,\hat{z}\cdot\hat{z}\right)$$

$$= \gamma \int_0^{2\pi} d\phi \int_0^R ds$$

$$= 2\pi R \gamma$$