Laplace's Eqn

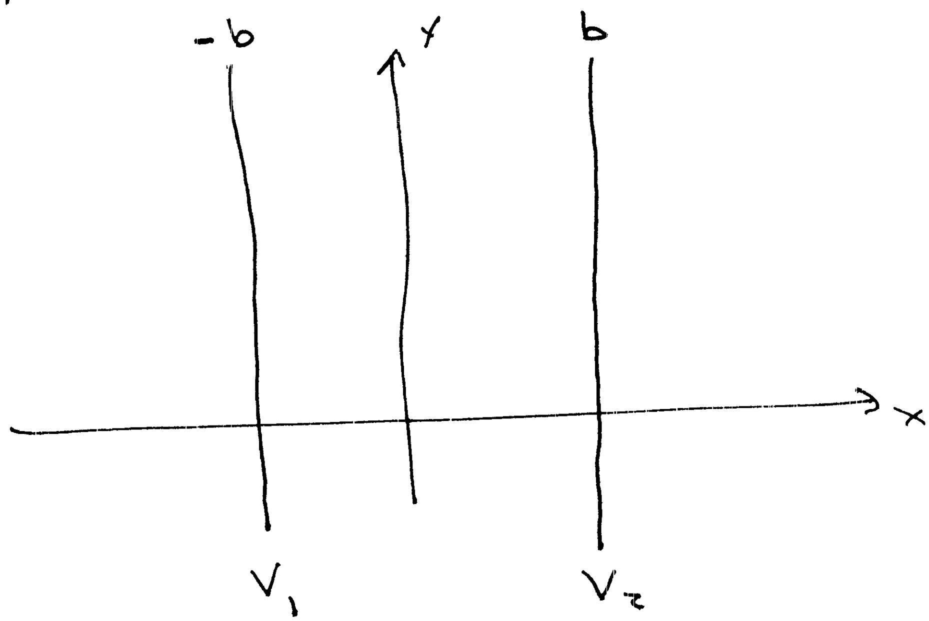
In general, we need the solution to Laplace's eqn in cartesian, cylindrical, and spherical coordinates.

There are lots of special cases.

Cartesian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

1D - Potential does not change in y or z direction. Field between two infinite plates held at potential V_1 and V_2



Since $\rho = 0$ between the plates Laplace's eqn applies.

$$\frac{\partial^2 V}{\partial x^2} = 0$$

Solutions $\quad V = 1, \quad V = x$

General Solution

$$V = a_1 + a_2 x \qquad a_i \text{ constant}$$

Apply BC

$$V(-b) = V_1 = a_1 + a_2(-b)$$

$$V(b) = V_2 = a_1 + a_2(b)$$

$$V_1 + V_2 = 2a_1 \qquad a_1 = \frac{V_1 + V_2}{2}$$

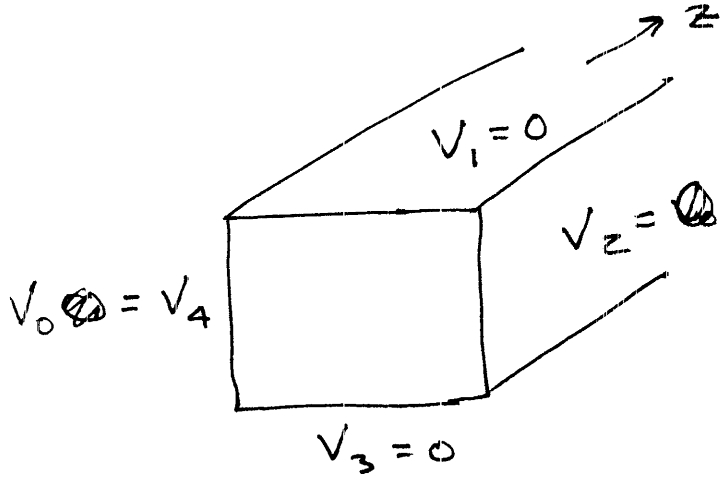
$$V_2 - V_1 = 2a_2 b$$

$$a_2 = \frac{V_2 - V_1}{2b}$$

Solution $\quad V(x,y,z) = \dfrac{V_1 + V_2}{2} + \dfrac{V_2 - V_1}{2b} x$

2D Cartesian

\- Potential does not change in z direction $\Longrightarrow$ infinitely long channel



For example,

$$V(b, y, z) = 0$$

$$V(0, y, z) = V_0$$

$$V(x, 0, z) = 0$$

$$V(x, b, z) = 0$$

Laplace's Eqn

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Separation of Variables

Propose a solution $V(x,y) = X(x)\,Y(y)$ that is a product of functions of one variable

- Does not always work
- Does not yield all solutions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} = 0$$

Divide by V.

$$\underbrace{\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}}_{= k^2} + \underbrace{\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}}_{= -k^2} = 0$$

Since each depend on different variables, each must be independently constant. Let that constant be $\pm k^2$.

The choice of which is positive or negative depends on the boundary conditions.

With my choice

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k^2$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k^2$$

$$\frac{d^2 X}{dx^2} - k^2 X = 0 \qquad \frac{d^2 Y}{dy^2} + k^2 Y = 0$$

Solutions

$$X(x) = \begin{matrix} e^{kx} \\ e^{-kx} \end{matrix} \qquad Y(y) = \begin{matrix} \sin kx \\ \cos kx \end{matrix}$$

The boundary conditions may not allow all values of k.

Impose y boundary conditions

$$Y(0) = 0 \qquad Y(b) = 0$$

$$Y(0) = 0 \implies Y(y) = \sin kx$$

$$Y(b) = 0 \implies \sin kb = 0$$

$$k = k_n \equiv \frac{n\pi}{b}$$ only allowed values.

At this point, any $e^{\pm k_n x} \sin k_n y$ solves Laplace's equation and the y boundary condition.

Our general solution is any sum of these functions

$$V(x,y) = \sum_{n=1}^{\infty} \sin k_n y \left(C_n^+ e^{k_n x} + C_n^- e^{-k_n x} \right)$$

<u>Impose x - boundary condition</u>

$$V(0,y) = V_0 = \sum_{n=1}^{\infty} \left(C_n^+ + C_n^- \right) \sin k_n y$$

$$V(b,y) = 0 = \sum_{n=1}^{\infty} \left(C_n^+ e^{k_n b} + C_n^- e^{-k_n b} \right) \sin k_n y$$

We need to solve for $C_n^{\pm}$; how do we pick these apart?

Some solutions to Laplace's equation under certain boundary conditions have some very useful properties.

Let $\{f_n\}$ be a set of solutions to $\nabla^2 f = 0$.

<u>Completeness</u> - The set $\{f_n\}$ is complete if any function f satisfying $\nabla^2 f = 0$ and the boundary conditions can be written as a linear combination of $\{f_n\}$

$$f(x) = \sum_{n=1}^{\infty} a_n f_n(x)$$

where a_n is constant.

We have already used this when we assumed the solution to our system could be written as the sum of the separated solutions.

Orthogonality

$$\int_0^b \sin k_n x \, \sin k_m x \, dx = \begin{cases} 0 & , \text{if } m \neq n \\ \frac{b}{2} & , \text{if } m = n \end{cases}$$

$$= \frac{b}{2} \delta_{mn}$$

Kronecker Delta

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

* The functions $\{\sin k_n x\}$ behave like orthogonal vectors.
* $\{\sin k_n x\}$ are linearly independent.

To extract the coefficients, a_n in

$$f(x) = \sum_n a_n f_n(x)$$

Multiply by $f_m(x)$ and use orthogonality.

For example, if

$$f(x) = \sum_n a_n \sin k_n x$$

$$\int_0^b \sin k_m x \, f(x) \, dx = \sum_n \int_0^b a_n \sin k_n x \sin k_m x \, dx$$

$$= \sum_n a_n \frac{b}{2} \delta_{nm} = \frac{b}{2} a_m$$

$$a_m = \frac{2}{b} \int_0^b \sin k_m x \, f(x) \, dx$$

For our series,

$$V(0,y) = V_0 = \sum_{n=1}^{\infty} (C_n^+ + C_n^-) \sin k_n y$$

Multiply by $\sin k_m x$ and integrate

$$\int_0^b V_0 \sin k_n y \, dy = \sum_{n=1}^{\infty} (C_n^+ + C_n^-) \underbrace{\int_0^b \sin k_m y \sin k_n y \, dy}_{b/2 \, \delta_{mn}}$$

$$= \frac{b}{2}(C_m^+ + C_m^-)$$

$$V_0 \int_0^b \sin k_m y \, dy = V_0 \int_0^b \sin \frac{m y \pi}{b} \, dy$$

$$= V_0 \frac{b}{m\pi} \int_0^{m\pi} \sin u \, du = \frac{V_0 b}{m\pi} (-\cos u)\Big|_0^{m\pi}$$

$$= \frac{V_0 b}{m\pi}(1 - \underset{\cos m\pi}{\cancel{m\pi}}) = \begin{cases} 0 & m \text{ even} \\ \dfrac{2V_0 b}{m\pi} & m \text{ odd} \end{cases}$$

$$C_m^+ + C_m^- = \begin{cases} 0 & m \text{ even} \\ \dfrac{4V_0}{m\pi} & m \text{ odd} \end{cases}$$

Now apply $V(b,y) = 0$ boundary condition

$$V(b,y) = 0 = \sum_{n=1}^{\infty} \left(C_n^+ e^{bk_n} + C_n^- e^{-bk_n} \right) \sin k_n y$$

<u>Multiply by $\sin k_m y$ and integrate</u>

$$0 = \sum_{n=1}^{\infty} \left(C_n^+ e^{bk_n} + C_n^- e^{-bk_n} \right) \int_0^b \sin k_m y \sin k_n y \, dy$$

$$= \frac{b}{2} \left(C_n^+ e^{bk_m} + C_n^- e^{-bk_m} \right) = 0$$

$$C_m^- = -C_m^+ e^{2bk_m}$$

$$C_m^+ + \left(-C_m^+ e^{2bk_m} \right) = \begin{cases} 0 & m \text{ even} \\ \dfrac{4V_0}{\pi m} & m \text{ odd} \end{cases}$$

$$C_m^+ = \begin{cases} 0 & m \ \cancel{\text{odd}} \text{ even} \\ \dfrac{4V_0}{\pi m(1 - e^{2k_m b})} & m\text{-odd} \end{cases}$$

Full Solution

$$V(x,y) = \sum_{n\,odd} \frac{4V_0 \sin k_n y}{\pi n(1 - e^{2k_n b})}\left(e^{k_m x} - e^{-k_m x} e^{2k_n b}\right)$$

From this we can calculate the field

$$\vec{E} = -\nabla V$$

The charge density at one of the faces, assuming conducting walls.

$$\sigma = \varepsilon_0 E_n = -\varepsilon_0 \frac{\partial V}{\partial n}$$

For example, the charge on the plane in the x-z plane

$$\sigma = \varepsilon_0 \, \hat{x} \cdot \vec{E}(0,y)$$

$$= -\varepsilon_0 \left. \frac{\partial V}{\partial x} \right|_0$$

$$= \sum_{n \text{ odd}} - \frac{4V_0 \sin k_n y}{n\pi(1-e^{2k_n b})} \left(k_m + k_m e^{2k_m b} \right)$$

Final Notes

For each coordinate system, there will be some special solutions that are sometimes useful that are not part of the complete set.

For 2D Cartesian,

$1, x, y$ are trivially solutions of $\nabla^2 f = 0$.

$\Longrightarrow$ These are the $k=0$ solutions

Laplace's Eqn - Cartesian - 3 dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

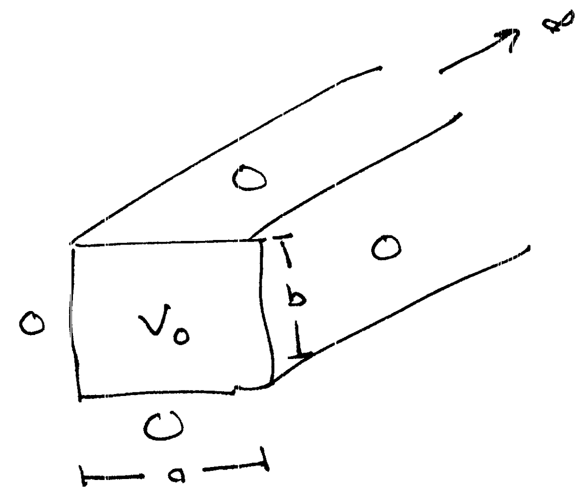
Separation $V = X(x)\,Y(y)\,Z(z)$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

$$\underset{-k^2}{\overset{\shortparallel}{}} \qquad \underset{-l^2}{\overset{\shortparallel}{}} \qquad \underset{w^2}{\overset{\shortparallel}{}}$$

$$w^2 = k^2 + l^2$$

Ex



Solutions

X after BC

$$X(x) = \sin k_n x \qquad k_n = \frac{n\pi}{a}$$

Y after BC

$$Y(y) = \sin l_m y \qquad l_m = \frac{m\pi}{b}$$

Z solution

$$Z(z) = e^{wz}, \quad e^{-wz}$$

can discard e^{wz} because it blows up as $z \to \infty$

$$V(x,y,z) = \sum_m \sum_n A_{nm} \sin k_n x \sin l_m y \, e^{-(k_n^2 + l_m^2)^{1/2} z}$$