Spherical Symmetry

Laplace's Eqn

$$\nabla^2 V = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Radial Symmetry - BC independent of θ, ϕ

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

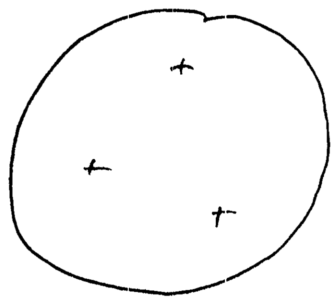
Let $V = r^n$ $\quad \frac{\partial V}{\partial r} = n r^{n-1}$

$$\frac{\partial}{\partial r}\left(r^2 n r^{n-1}\right) = n \frac{\partial}{\partial r}\left(r^{n+1}\right) = 0$$

$n = -1$

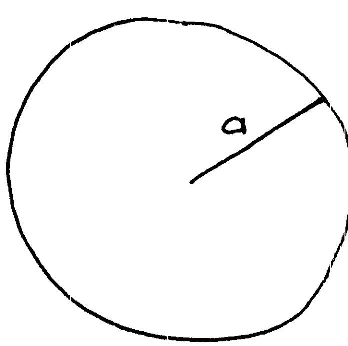
Solutions $\quad V = 1, \frac{1}{r}$

Ex Spherical Volume Charge



$\Longrightarrow$ Cannot solve because $\rho \neq 0$ in the volume

Ex Thin ~~sphere~~ shell of charge of radius a and charge density σ.



The regions inside and outside the shell satisfy Laplace's eqn.

General Solution

<u>inside</u> $\quad V_i = A_1 + \dfrac{B_1}{r}$

<u>outside</u> $\quad V_o = A_2 + \dfrac{B_2}{r}$

Boundary Conditions

I. $V_o(\infty) = 0 \implies A_2 = 0$

II. $V_i(0)$ finite $\implies B_1 = 0$

III. Continuous at a

$$V_i(a) = A_1 = V_o(a) = \frac{B_2}{a}$$

$$B_2 = a A_1$$

IV. Derivative satisfies electrostatic BC.

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = \frac{-\sigma}{\varepsilon_0} \qquad (2.36)$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a = -\frac{B_2}{a^2} \qquad \left.\frac{\partial V_i}{\partial r}\right|_a = 0$$

$$-\frac{B_2}{a^2} = -\frac{\sigma}{\varepsilon_0}$$

$$B_2 = \frac{a^2\sigma}{\varepsilon_0} \implies \cancel{A_1 = \frac{a^3\sigma}{\varepsilon_0}}$$

~~Full~~ $$A_1 = \frac{B_2}{a} = \frac{a\sigma}{\varepsilon_0}$$

Full Solution

$r < a \qquad V_i = \dfrac{a\sigma}{\varepsilon_0}$

$r > a \qquad V_o = \dfrac{a^2\sigma}{\varepsilon_0 r} = \dfrac{4\pi a^2\sigma}{4\pi\varepsilon_0 r} = \dfrac{Q}{4\pi\varepsilon_0 r}$

Field

$$\vec{E}_i = -\nabla V_i = 0$$

$$\vec{E}_o = -\nabla V_o = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Laplace's Eqn - Spherical Coordinates - Azimutal Symmetry (independant of ϕ)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

Separation

$$V(r,\theta) = R(r)\,\Theta(\theta)$$

Solutions $1, \frac{1}{r}, \ln\left(\tan \theta/2\right)$

$$r^n P_n(\cos\theta) \qquad r^{-(n+1)} P_n(\cos\theta)$$

Legendre Polynomial $P_n(x)$

$$P_0 = 1 \qquad P_1 = x \qquad P_2 = \frac{1}{2}(3x^2 - 1)$$

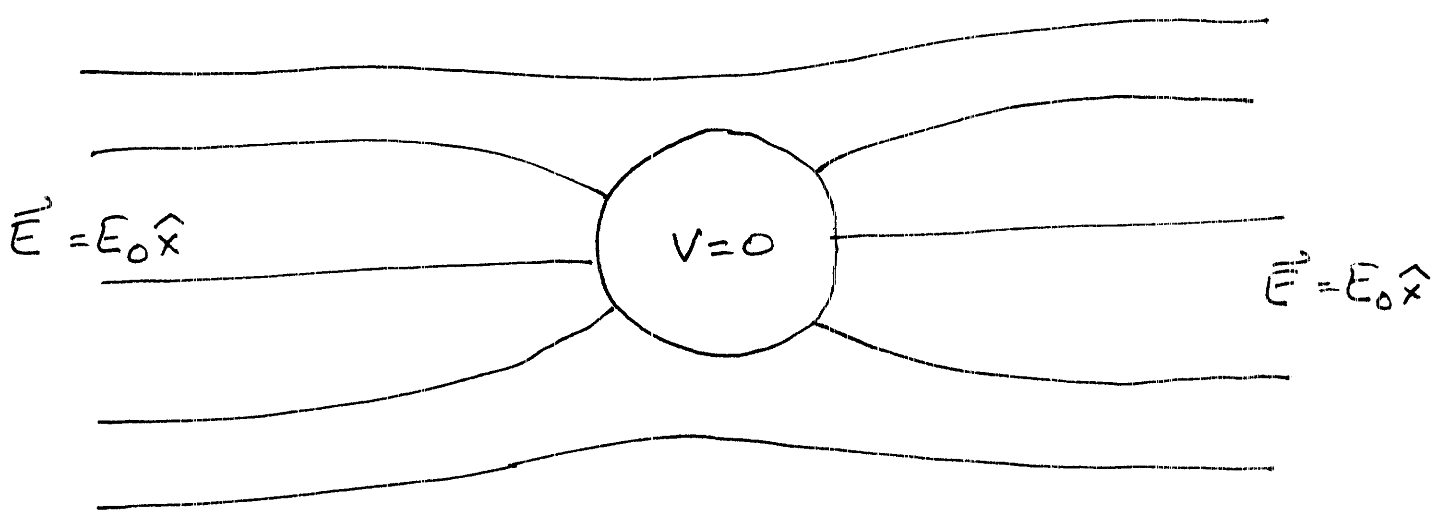
Orthogonality

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1}\, \delta_{nm}$$

-or-

$$\int_{-1}^{1} P_n(\cos\theta)\, P_m(\cos\theta)\, d(\cos\theta) = \frac{2}{2n+1}\, \delta_{nm}$$

<u>Ex</u> Conducting Sphere in Uniform Field

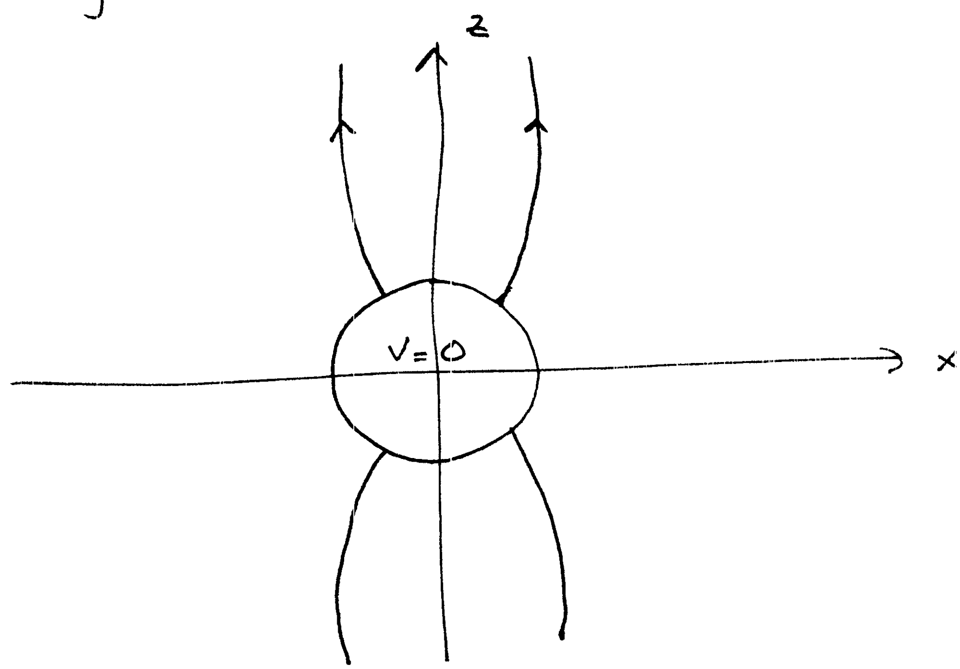


<u>Boundary Conditions</u> $\quad Q_{sphere} = 0$

$V_{sphere} = 0 \qquad \vec{E} \rightarrow E_0 \hat{x}$ at $x \rightarrow \pm\infty$.

<u>Solution</u> Spherical problem, does not depend on ϕ.

Exchange axes, $x \rightarrow z$



Write Boundary condition in spherical coordinates

$$\vec{E} = E_0 \hat{z} \implies V \rightarrow -E_0 z + C$$

From back cover $z = r\cos\theta$

$$V \rightarrow -E_0 r\cos\theta + C$$

Now, write the BC usefully,

$$P_1(x) = x \implies P_1(\cos\theta) = \cos\theta$$

$$V \to -E_0 r P_1(\cos\theta)$$

<u>General Solution</u> $(r > a)$

$$V(r,\theta) = \sum_n A_n r^{-(n+1)} P_n(\cos\theta) + B_n r^n P_n(\cos\theta)$$

<u>Look for explosions</u>, at $r \to \infty$, $r^n \to \infty$
so only B_1 survives. We need this to satisfy $V \to -E_0 r P_1(\cos\theta)$. The B_1 term is $B_1 r P_1(\cos\theta)$ so $B_1 = -E_0$

$$V(r,\theta) = \sum_n A_n r^{-(n+1)} P_n(\cos\theta) - E_0 r P_1(\cos\theta)$$

Since $r > a$, there are no explosions in the $r^{-(n+1)}$ terms.

Apply $r = a$ BC

$$V(a,\theta) = 0 = \sum_n A_n a^{-(n+1)} P_n(\cos\theta) - E_o a P_1(\cos\theta)$$

Method I – P_i are orthogonal functions, we can treat them like orthogonal vectors. So the above is like

$$\vec{A} = (1\hat{x} + 2\hat{y} + 3\hat{z}) + 2\hat{x}$$

$$= (3\hat{x} + 2\hat{y} + 3\hat{z})$$

So $A_n = 0$, $n \geq 2$

$$A_1 a^{-(1+1)} - E_o a = 0$$

$$A_1 = a^3 E_o$$

$$V(r,\theta) = \frac{a^3 E_o}{r^2} P_1(\cos\theta) - E_o r P_1(\cos\theta)$$

$$= \left(\frac{a^3 E_o}{r^2} - E_o r\right) \cos\theta$$

Method II Multiply both sides by $P_m(\cos\theta)$ and integrate

$$\int_{-1}^{1} 0\, P_m(\cos\theta)\, d(\cos\theta) = \sum_n A_n a^{-(n+1)} \int_{-1}^{1} P_m(\cos\theta)\, P_n(\cos\theta)\, d(\cos\theta)$$

$$+ \int_{-1}^{1} (-E_0 a)\, P_m(\cos\theta)\, P_1(\cos\theta)\, d\cos\theta$$

If $m \geq 2$,

$$0 = A_m a^{-(m+1)} \cdot \frac{2}{2m+1} \implies A_m = 0$$

If $m=1$,

$$0 = \frac{A_1}{a^2} \cdot \frac{2}{2\cdot 1+1} - E_0 a \cdot \frac{2}{2\cdot 1+1}$$

$$A_1 = E_0 a^3$$

If all went well, $\vec{E} \perp$ surface and

$Q = 0$. Check.

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} \right)$$

$$= \left(E_0 \cos\theta + \frac{2 E_0 a^3 \cos\theta}{r^3} \right) \hat{r}$$

$$- \left(\frac{E_0 r \sin\theta}{r} - \frac{E_0 a^3 \sin\theta}{r^3} \right) \hat{\theta}$$

$$\vec{E}(a) = 3 E_0 \cos\theta \, \hat{r}$$

<u>Surface Charge</u>

$$\phi = \vec{E}_0 \cdot \hat{r} - \underbrace{\vec{E}_i \cdot \hat{r}}_{0} = \frac{\sigma}{\varepsilon_0}$$

$$3 E_0 \cos\theta = \sigma / \varepsilon_0$$

$$\sigma = 3 \varepsilon_0 E_0 \cos\theta$$

Total Charge on Sphere

$$Q = \int \sigma \, da$$

$$da = (R\,d\theta)(R \sin\theta \, d\phi)$$

$$Q = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, (R^2 \sin\theta)(3 E_0 \varepsilon_0 \cos\theta)$$

$$= 6\pi E_0 \varepsilon_0 R^2 \underbrace{\int_0^{\pi} \sin\theta \cos\theta \, d\theta}_{0} \qquad = 0$$

since $\sin\theta \cos\theta = \dfrac{\sin 2\theta}{2}$

Full General Solution - Spherical Coordinates

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-(l+1)}\right] Y_l^m(\phi,\theta)$$

Spherical Harmonics

$$Y_l^m = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \, P_l^m(\cos\theta)\, e^{im\phi}$$

Associated Legendre Polynomial

Orthogonality

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; Y_{l'}^{*m'} \, Y_l^m = \delta_{ll'} \delta_{mm'}$$

There are two trivial angular solutions.

$V = \ln\left(\tan \theta/2\right)$ when BC only depend on θ

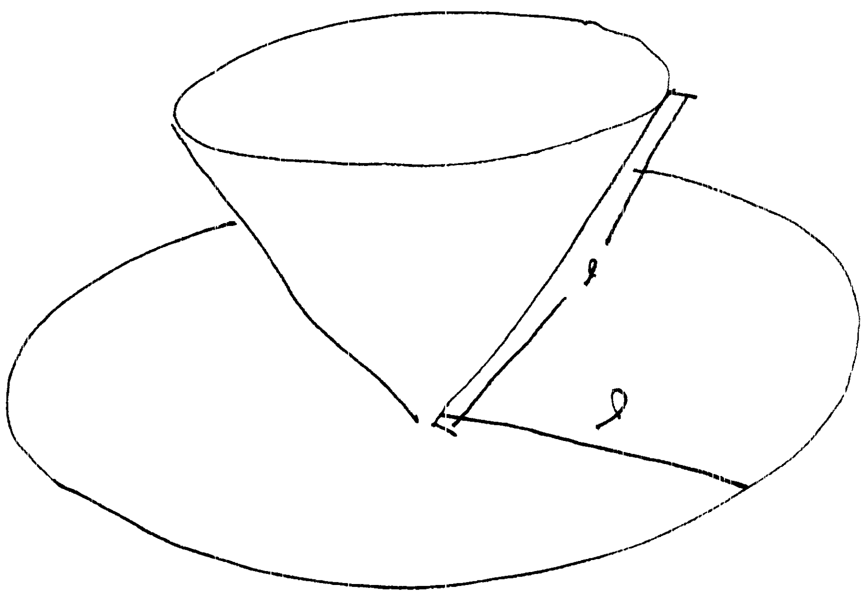
and

$V = \phi$

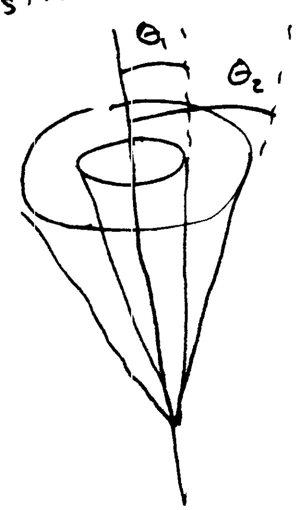
when BC only depend on ϕ.

I give an example of using the first next. We will find the same trivial solution for ϕ in cylindrical and an example will be given there.

Ex Compute field / capacitance of $45°$ cone, height $\frac{\ell}{\sqrt{2}}$ above circular disk of radius ℓ. No fringing (big assumption). Let disk be held at V_0, cone at 0.



Same type of system as



Try solution with no r, ϕ dependence

$$V(\theta) = A \ln\left(\tan \theta/2\right) + B$$

Boundary conditions

$$V(45^\circ) = 0$$

$$V(90^\circ) = V_0$$

$\Rightarrow$ We will automatically satisfy $\vec{E} \parallel \hat{n}$ because the surfaces are equipotentials.

$$V(45) = A \ln\left(\tan \frac{\pi}{8}\right) + B = 0$$

$$V(90) = A \ln\left(\tan \frac{\pi}{4}\right) + B$$

$$= A \ln(1) + B = B = V_0$$

$$A = \frac{-V_0}{\ln(\tan \pi/8)}$$

$$V(\theta) = -V_0 \left(\frac{\ln(\tan \theta/2)}{\ln(\tan \pi/8)} - 1 \right)$$

The electric field is

$$\vec{E} = -\nabla V = -\frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta}$$

$$= \frac{-V_0}{\ln(\tan \pi/8)} \frac{\cancel{\sin\theta}}{r \sin\theta} \hat{\theta} = \frac{-V_0}{\gamma \, r \sin\theta} \hat{\theta}$$

$$\gamma \equiv \ln(\tan \pi/8)$$

The charge density on the flat disk is

$$\sigma = -\frac{V_0}{\gamma r} \varepsilon_0$$ (using Gaussian Pillbox)

The total charge on the disk is

$$Q = \int_0^{2\pi} \int_0^{l} r \, d\theta \, dr \, \sigma$$

$$= -\frac{V_0}{\gamma} \int_0^{2\pi} \int_0^{l} d\theta \, dr = -\frac{V_0 \, 2\pi l}{\gamma}$$

- Capacitance

$$C = \left| \frac{Q}{V_0} \right| = \frac{2\pi \ell}{\ln\left(\tan \frac{\pi}{8}\right)}$$