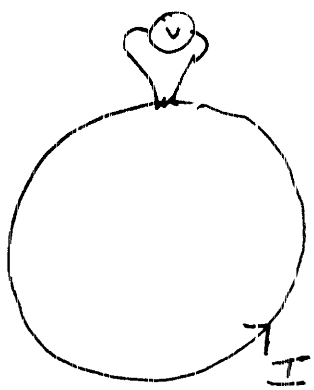
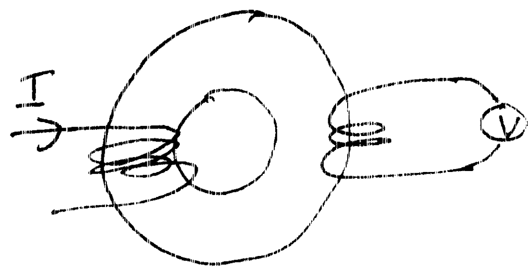
Magnetic Energy

If a current flows through a circuit with self-inductance or mutual inductance, a voltage will appear across the device.





By Faraday's Law, the emf is related to the current by

$$\text{emf} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt} LI = -L\frac{dI}{dt}$$

$$= -\frac{d\Phi_m}{dt} = -\frac{d}{dt} MI = -M\frac{dI}{dt}$$

This allows us to calculate the amount of work needed to establish the current.

$$W = \int P\,dt = \int I \Delta V\,dt$$

$$= \int I \left(L \frac{dI}{dt} \right) dt$$

$$= L \int I\,dI = \frac{1}{2} L I^2$$

= Energy stored in the fields since there are no intrinsic losses in an inductor.

Energy Stored in Inductor

$$U = \frac{1}{2} L I^2$$

This energy can be written in terms of the potential using

$$\Phi_m = \int_c \vec{A} \cdot d\vec{l}$$

$$\Phi_m = L I$$

$$U = \frac{1}{2} L \Phi_m = \frac{1}{2} I \oint \vec{A} \cdot d\vec{l}$$

$$= \frac{1}{2} \oint_C (\vec{A} \cdot \vec{I})\, dl$$

or in terms of the current density

$$U = \frac{1}{2} \int_V \vec{A} \cdot \vec{J}\, d\tau$$

Compare this to the energy in the electric field

$$U = \frac{1}{2} \int \rho V \, d\tau$$

We also wrote an energy density in terms of the electric field as $u = \frac{1}{2} \varepsilon_0 E^2$.

Let's find the corresponding magnetic version.

Consider an infinitely long solenoid carrying current I. The energy stored in a length l of the inductor is

$$U = \frac{1}{2} L I^2 = \frac{1}{2}\left(\mu_0 n^2 A l\right) I^2$$

$$= \frac{1}{2\mu_0} \cdot \left(\mu_0 n I\right)^2 A l$$

$$= \frac{B^2}{2\mu_0} \cdot A l$$

Al is the volume inside the inductor, therefore the energy density of the magnetic field in the inductor is

$$u = \frac{U}{V} = \frac{U}{Al} = \frac{B^2}{2\mu_0}$$

Since magnetic field is magnetic field, this must be the energy density of any magnetic field.

Now, fill the solenoid with a linear magnetic material. The resulting magnetic field is

$$B = \mu_r B_0$$

The inductance becomes

$$L = \frac{\Phi_m}{I} = \frac{N A \mu_0 \mu_r n I}{I}$$

$$= \mu_0 \mu_r n^2 A l$$

The inductance is increased by μ_r.

The energy may be calculated from $U = \frac{1}{2} L I^2$.

$$U = \frac{1}{2} L I^2 = \frac{1}{2} \left(\mu_0 \mu_r n^2 A l \right) I^2$$

$$= \frac{1}{2} \left(\mu_r \mu_0 n I \right) \left(n I \right) A l$$

Recall $\mu_0 \mu_r \vec{H} = \vec{B}$

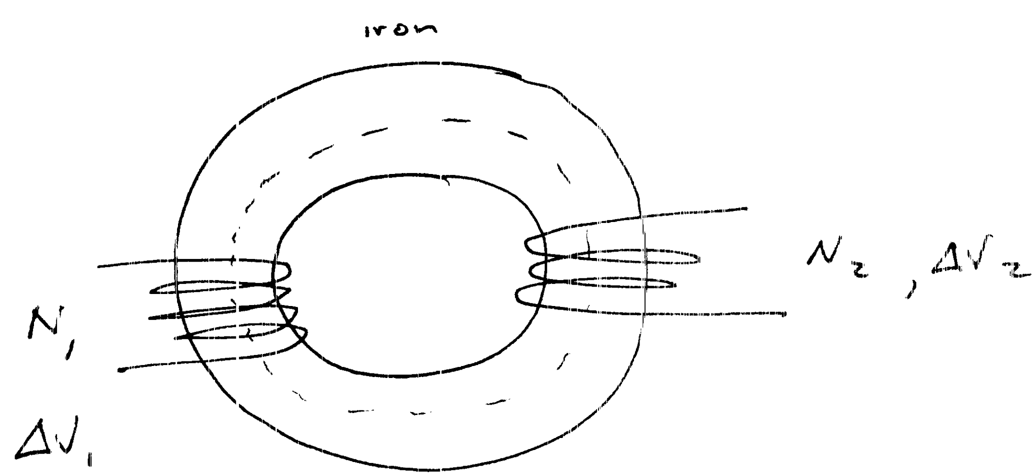
$$|\vec{H}| = n I \qquad \vec{B} = \mu_r \mu_0 n I$$

$$U = \frac{1}{2} \vec{B} \cdot \vec{H}$$

Compare this with the energy density in an electric field in a material

$$u = \frac{1}{2}\vec{D}\cdot\vec{E}$$

Transformers - Consider a device formed of two coils wound on a closed iron circuit.



The field lines are trapped in the iron (somewhat) so every field line passing through loop 1 passes through loop 2. Therefore the flux through any loop is the same ϕ_0. We can then calculate the flux

$$\Phi_1 = N_1\phi_0 \qquad \Phi_2 = N_2\phi_0$$

The ratio of the voltages can be calculated

$$\frac{\Delta V_2}{\Delta V_1} = \frac{-N_2 \frac{d\phi_0}{dt}}{-N_1 \frac{d\phi_0}{dt}} = \frac{N_2}{N_1}$$

Commercial transformers are built to be fairly efficient so to a good approximation power in = power out.

Each time the transformer goes through an AC cycles, the iron is driven through its hysteresis loop an is heated by the area $\frac{1}{2}\vec{B}\cdot\vec{H}$ under the curve.

Real transformers work to prevent this heating by using a laminar ferrous material.