Magnetic Force

The magnetic force, the Lorentz force, exerted on a moving charge by a magnetic field is

$$\vec{F} = q\vec{v} \times \vec{B}$$

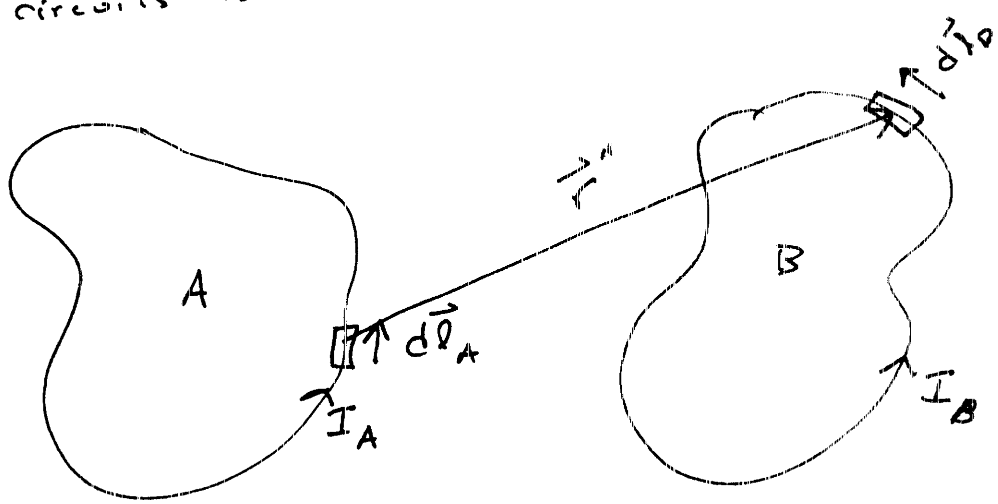
if the moving charge is part of an electrical current

$$\vec{F} = \int_c I\, d\vec{l}\,' \times \vec{B}(\vec{r}\,')$$

$$= \int_s \vec{K}' \times \vec{B}(\vec{r}\,')\, da'$$

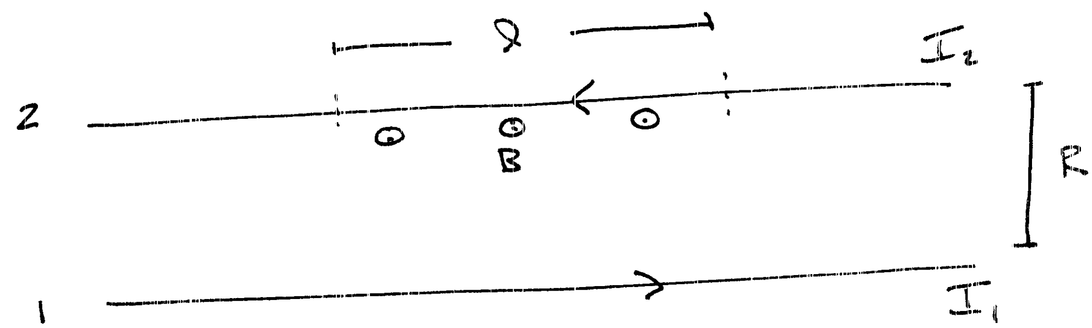
$$= \int_v \vec{J}' \times \vec{B}(\vec{r}\,')\, d\tau'$$

The analog of Coulomb's law for two closed electric circuits is



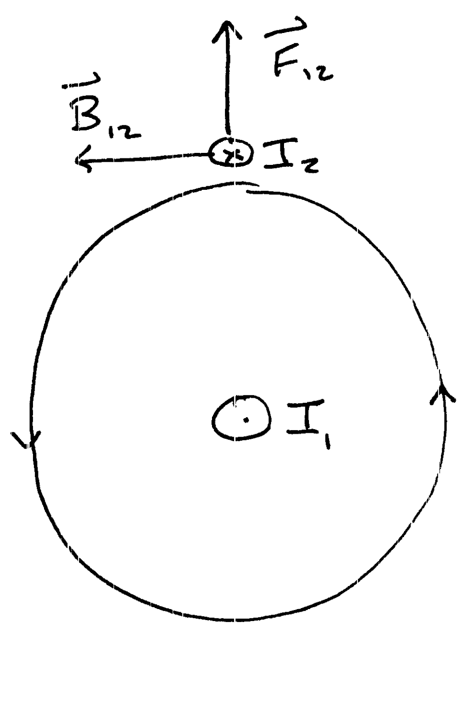
$$\vec{F}_{AB} = \frac{\mu_0 I_A I_B}{4\pi} \oint_{C_A} \oint_{C_B} d\vec{l}_B \times \left(\frac{d\vec{l}_A \times \hat{r}''}{r''^2} \right)$$

Ex Force per unit length one long wire exerts on another wire.



Compute force bottom wire exerts on a length l of the top wire.

Side View



The magnetic field of the bottom wire at the top wire is

$$B_{12} = \frac{\mu_0 I_1}{2\pi R}$$

The magnetic force is

$$|\vec{F}_{12}| = I_2 \int d\vec{l}_2 \times \vec{B}_{12}$$

but $d\vec{l}_2 \perp \vec{B}_{12}$ so $|d\vec{l}_2 \times \vec{B}_{12}| = dl_2 B_{12}$

$$|\vec{F}_{12}| = I_2 B_{12} \int_C dl_2 = I_2 B_{12}\, l$$

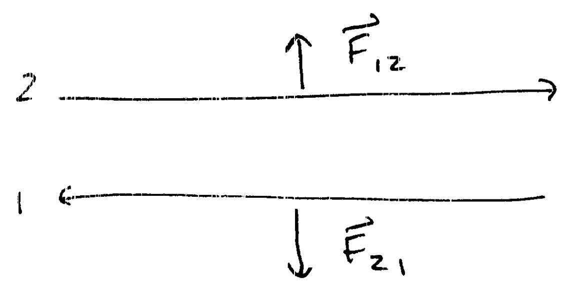
$$\frac{|\vec{F}_{12}|}{l} = \frac{\mu_0 I_1 I_2}{2\pi R}$$

For house wiring, $I_{max} = I_1 = I_2 = 20A$

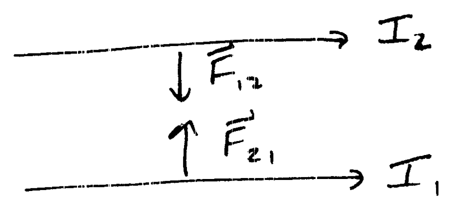
$R \cong 1cm$

$$\frac{|\vec{F}_{12}|}{l} = \frac{(4\pi \times 10^{-7} Tm/A)(20A)^2}{(2\pi)(0.01m)} = 8 \times 10^{-3} N/m$$

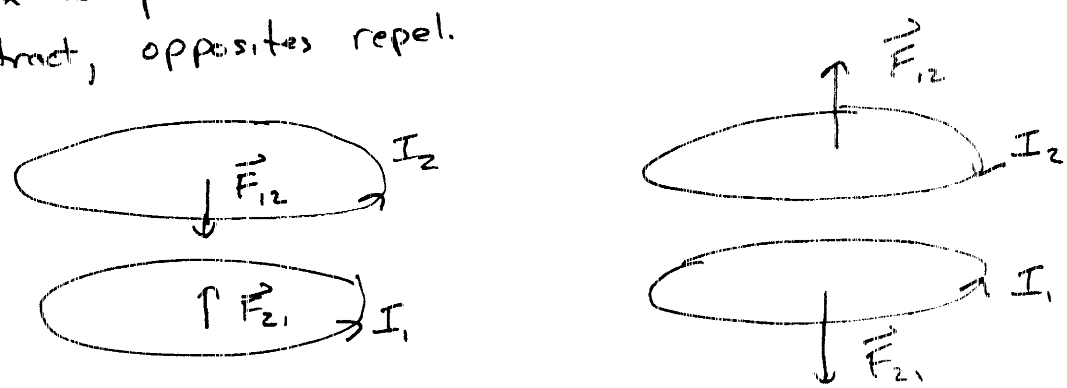
Note, currents flowing in the opposite direction repel.



Currents flowing in the same direction attract.



We can wrap these wires into rings, like oriented currents attract, opposites repel.



Note, if the rings are close together compared to their radius, $\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi R}$ is not a bad approximation.

Ex Force one turn exerts on neighboring turn in sewer pipe solenoid. $N = 79$ $L = 79\text{cm}$

Turn spacing $= \dfrac{L}{N} \equiv d = 1\text{cm}$ $R = 5\text{cm}$

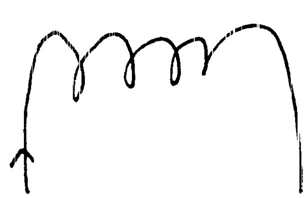
Force per turn $= \left(\dfrac{\mu_0 I^2}{2\pi d}\right) \cdot 2\pi R$ attractive

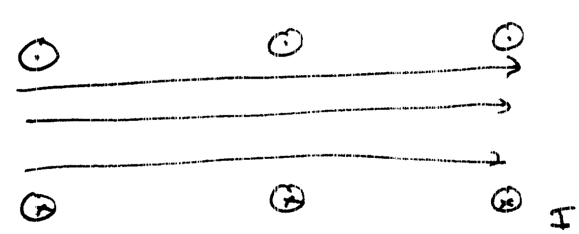
$$= \mu_0 I^2 \frac{R}{d}$$

$$= \left(4\pi \times 10^{-7}\,\text{Tm/A}\right)\left(20\,\text{A}\right)^2 \left(\frac{5\text{cm}}{1\text{cm}}\right)$$

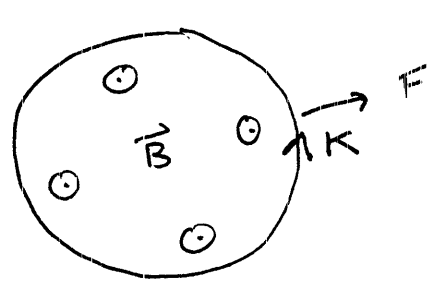
$$= 8\pi \times 10^{-4}\,\text{N}$$ attractive

Ex Pressure on long solenoid





End View



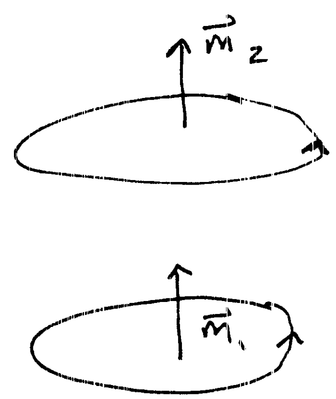
Magnetic pressure (static current assumed)

$$P = \left| \vec{K} \times \tfrac{1}{2}\left(\vec{B}_{inside} + \vec{B}_{outside}\right)\right|$$

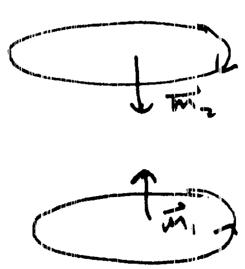
$$= \tfrac{1}{2}(K)(\mu_0 K) = \tfrac{1}{2}\mu_0 K^2$$

$$= \tfrac{1}{2}\mu_0 \left(\frac{N}{l}\right)^2 I^2 \quad \underline{\text{outward}}$$

But current loops have magnetic moments

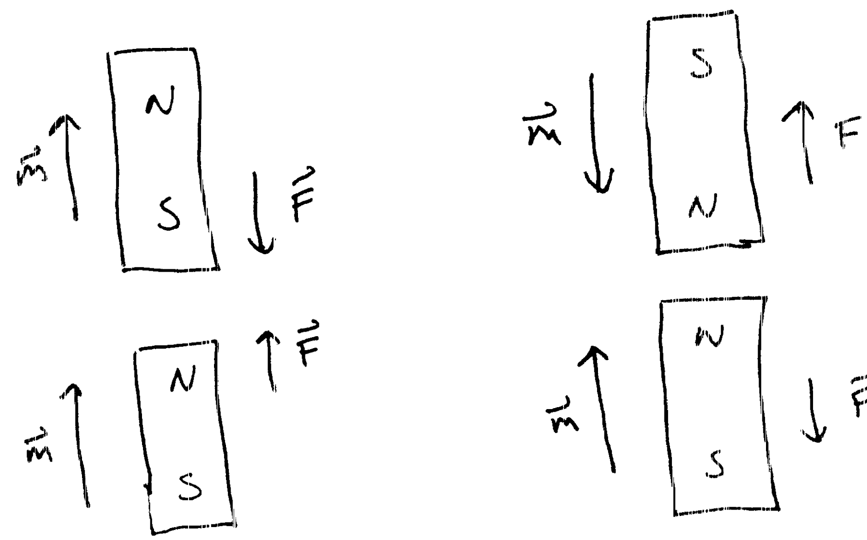


so aligned moments attract.



anti-aligned moments repel.

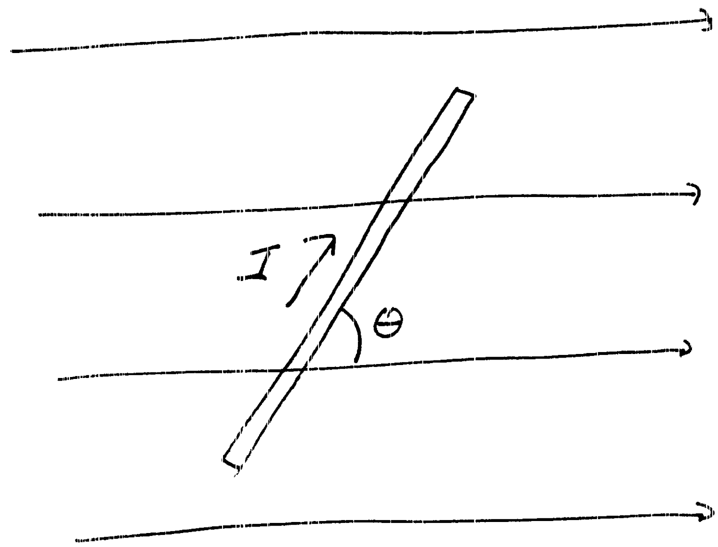
Label the tail of a moment vector South, the head North.



Like poles repel, Opposite poles attract.

$\Rightarrow$ All magnetic fields are produced by moving charge, there must be the equivalent of a set of current loops in magnetic materials.

<u>Ex</u> Force on straight wire at angle θ to uniform magnetic field. Length of wire ℓ.



$$\vec{F} = I \int_{wire} d\vec{\ell}' \times \vec{B}(\vec{r}')$$

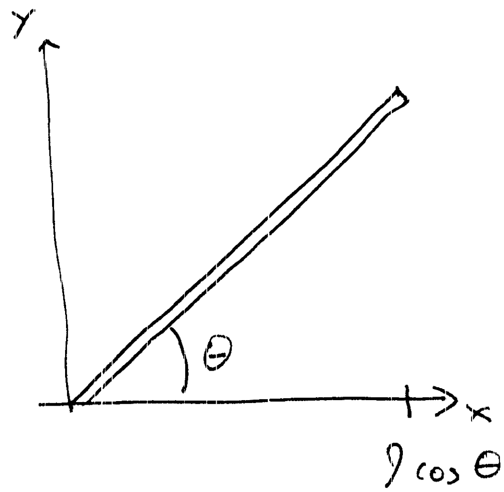
The force is into the page by the right-hand rule.

$$|d\vec{\ell}' \times \vec{B}(\vec{r}')| = d\ell' \, B(\vec{r}') \sin\theta$$

$$\vec{F} = I B \sin\theta \int_{wire} d\ell' \quad \text{into the page}$$

$$= I \ell B \sin\theta \quad \text{into the page.}$$

Let's work out the math for real



The wire is at $y' = \tan\theta\, x'$, $dy' = \tan\theta\, dx'$

$$d\vec{l}' = dx'\hat{x} + dy'\hat{y}$$

$$= dx'\hat{x} + \tan\theta\, dx'\hat{y}$$

$$\vec{B} = B_0\hat{x}$$

$$d\vec{l}'\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & \tan\theta\, dx' & 0 \\ B_0 & 0 & 0 \end{vmatrix}$$

$$= -B_0\tan\theta\, dx'\,\hat{z}$$

$$x \in [0, l\cos\theta]$$

$$\vec{F} = I \int d\vec{l}' \times \vec{B}$$

$$= -IB_0 \tan\theta\, \hat{z} \int_0^{l\cos\theta} dx'$$

$$= -IB_0 \tan\theta\, \hat{z}\; l\cos\theta \;=\; -IB_0\, l \sin\theta\, \hat{z}$$

<u>Ex</u> A square loop carrying current I with sides a is in a non-uniform magnetic field $\vec{B} = \frac{B_0}{a} x \hat{z}$.

The loop lies in the x-y plane, ~~centered at the origin~~ with one edge along y axis.

~~What current~~

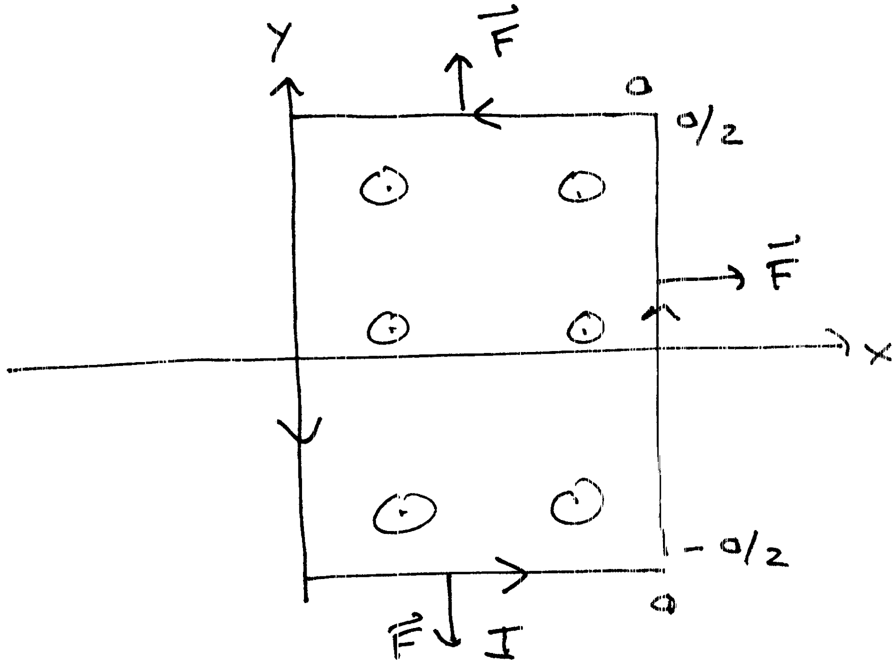
<u>Is this a valid magnetic field?</u>

$$\nabla \cdot \vec{B} = \frac{B_0}{a} \frac{\partial x}{\partial z} = 0$$ ✓ Yes

<u>What current density produced the field?</u>

$$\nabla \times \vec{B} = \mu_0 \vec{J} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{B_0}{a}x \end{vmatrix} = -\frac{B_0}{a}\hat{y} = \mu_0 \vec{J}$$

$$\vec{J} = \frac{-B_0}{\mu_0 a}\hat{y}$$



Evidently the force on the top and bottom cancel.

The force on the left side is zero, because $B=0$.

The force on the right side is

$$\vec{F} = I \int d\vec{l}' \times \vec{B}(\vec{r}')$$

$$d\vec{l}' = dx' \hat{y} \qquad \vec{B}(\vec{r}') = B_0 \hat{z}$$

$$d\vec{l}' \times \vec{B} = dx' B_0 \hat{x}$$

$$\vec{F} = I B_0 \hat{x} \int dl' = I B_0 a \hat{x}$$