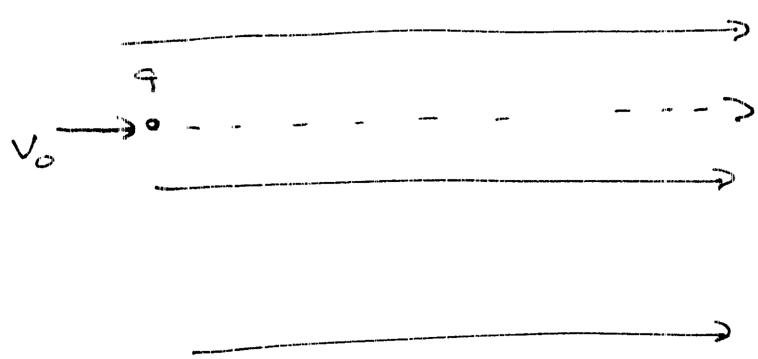
Electromagnetic Force

The electromagnetic force on a charged particle q moving with velocity $\vec{v}$ is

$$\vec{F} = q\vec{E} + q\,\vec{v}\times\vec{B}$$

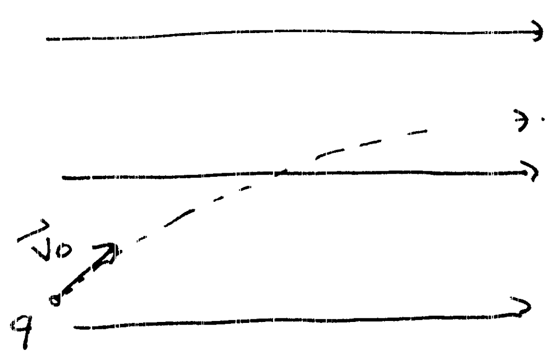
Let's compare the electric and magnetic forces

$$\vec{E} = E_0\,\hat{x}$$



$$x(t) = x_0 + v_0 t + \tfrac{1}{2} a t^2$$

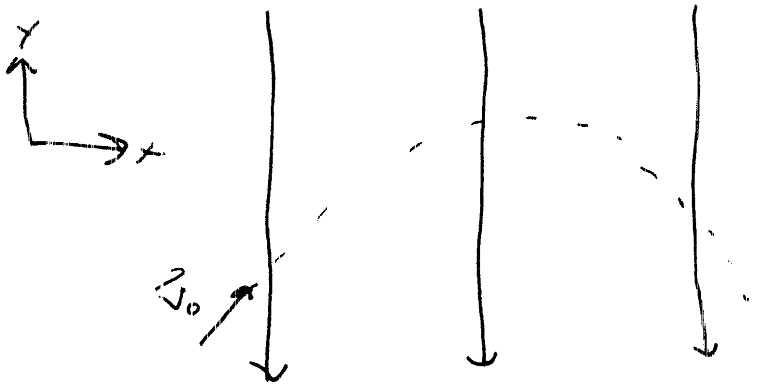
$$a = \frac{qE}{m}$$



$$x(t) = x_0 + v_{0x}\,t + \tfrac{1}{2} a t^2$$

$$y(t) = y_0 + v_{0x}\,t$$

$$\vec{E} = -E_0\hat{y}$$



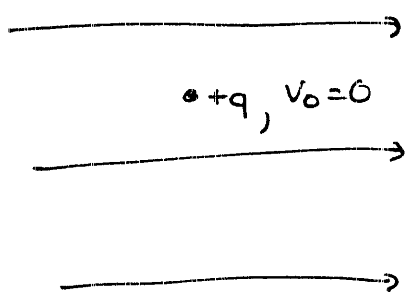
$$x(t) = x_0 + v_{0x}t$$

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}a_y t^2$$

$$a_y = \frac{qE_0}{m}$$

Naturally, all forces change direction if $q \to -q$

Now, consider a magnetic field. If a particle is released in the field



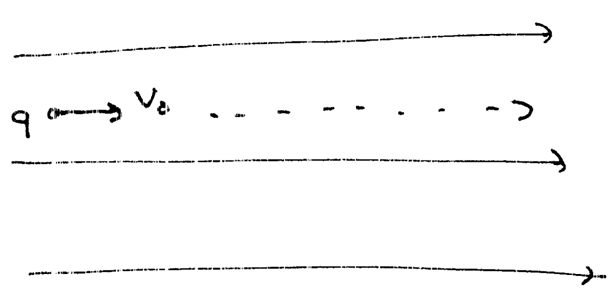
$$\vec{B} = B_0\hat{x}$$

The particle just sits there. $x(t) = x_0$ $y(t) = y_0$

If $\vec{v} = 0$, $\vec{F}_m = 0$.

Now, suppose the particle has initial velocity parallel to the field. $\vec{v}_0 = v_0\hat{x}$

$\vec{F}_m$ still is zero



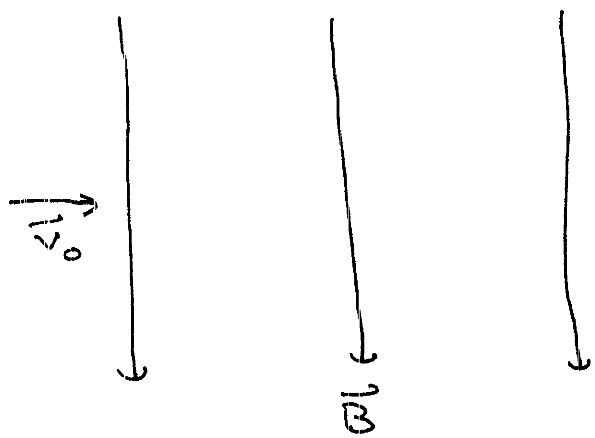
$$x(t) = x_0 + v_0 t$$

$$y(t) = y_0$$

No acceleration

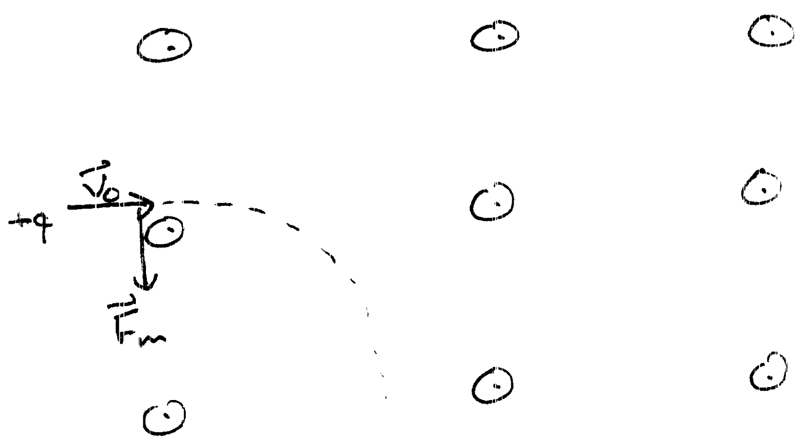
Now, fire a charge into a magnetic field with

$\vec{B} \perp \vec{v}_0$



Initial force is into the page.

<u>Different View</u>



Since $\vec{F} = q\vec{v} \times \vec{B}$, $\vec{F} \perp \vec{v}$, $\vec{F} \perp \frac{d\vec{r}}{dt}$

$\Longrightarrow \vec{F} \perp d\vec{r}$

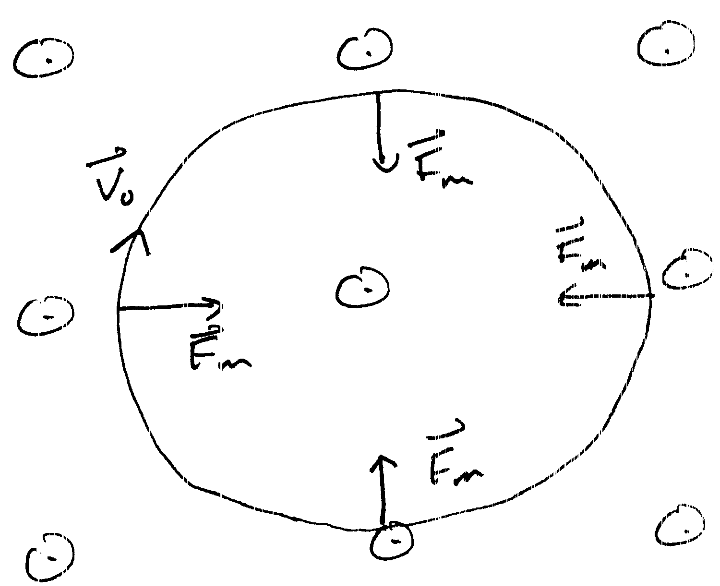
but Work $= \int \vec{F} \cdot d\vec{r} \Longrightarrow$ Work $= 0$

- The magnetic field does no work.
- The magnetic field cannot change the kinetic energy of the particle.

  $\Rightarrow$ Magnetic field can change direction but not speed.

For the particle we are considering, the force is in the plane of the page so $\vec{B} \perp \vec{v}$ and $F_m = q v_0 B$ = constant. The particle moves under the influence of a force of constant magnitude that acts $\perp$ to its velocity.

$\Rightarrow$ It moves in a circle.



The acceleration of anything moving in a circle is

$$a_c = \frac{v^2}{r} \Rightarrow \text{centripetal acceleration}$$

The centripetal acceleration can be written in terms of the angular velocity ω, $v = \omega r$

$$a_c = r\omega^2$$

$$\omega \equiv \frac{d\phi}{dt}$$

$$|F_m| = m a_c = \frac{mv^2}{r} \text{ (Newton II)}$$

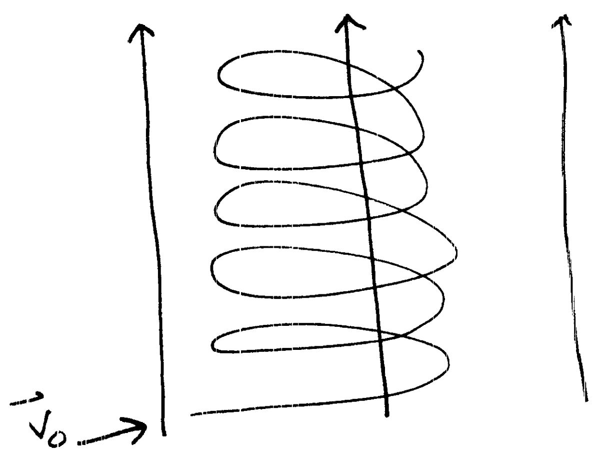
$$= m r \omega^2 = q v B = q \omega r B$$

$$\Rightarrow \quad r = \frac{mv}{qB} \quad \text{or} \quad \omega = \frac{qB}{m}$$

<u>Cyclotron Frequency</u> $\omega \equiv \dfrac{qB}{m}$

Now suppose the particle has some initial velocity ∥ to B. That velocity is not changed by B, so the particle travels in a helix.

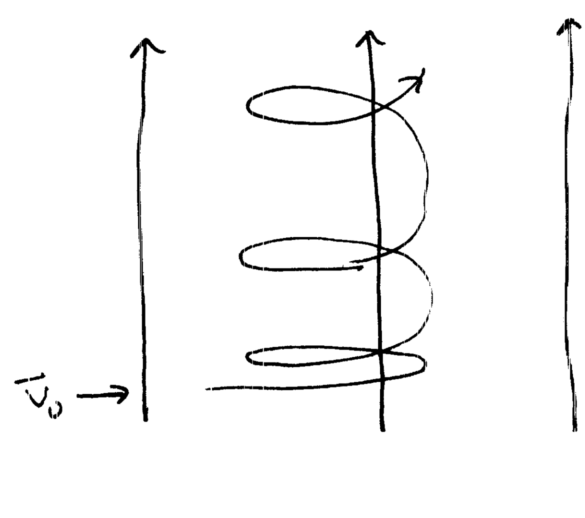
$\vec{B}$, $\vec{E}=0$



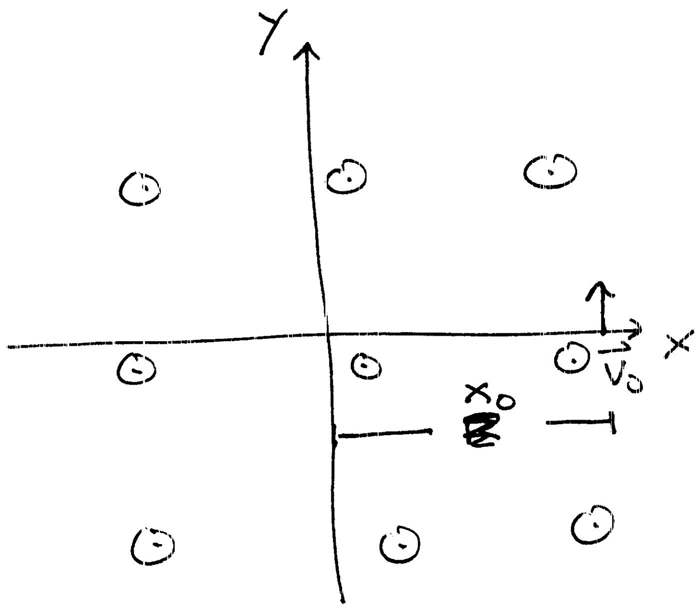
The wraps are equally spaced.

Now introduce an electric field ∥ to $\vec{B}$. The particle accelerates in the direction of $\vec{B}$, but the velocity ⊥ to B has a constant magnitude, so the particle travels in a helix of constant size with increasing spacing between wraps.

$\vec{B}=B_0\hat{y}$, $\vec{E}=E_0\hat{y}$



Let's do the mechanics properly



$$\vec{B} = B_0 \hat{z} \qquad \vec{r}_0 = (R, 0, 0) \qquad \vec{v}_0 = (0, v_0, 0)$$

<u>Newton's II Law</u>

$$\vec{F} = q \vec{v} \times \vec{B} = m \vec{a}$$

$$\vec{F} = q \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B_0 \end{vmatrix}$$

$$= q v_y B_0 \hat{x} - q v_x B_0 \hat{y}$$

Equations of Motion EOM

$$m a_x = q v_y B_0 = m \frac{d v_x}{dt}$$

$$m a_y = - q v_x B_0 = m \frac{d v_y}{dt}$$

using $\omega = \frac{q B_0}{m}$

$$\frac{d v_x}{dt} = \omega v_y \qquad \frac{d v_y}{dt} = - \omega v_x$$

Differentiate + Substitute

$$\frac{d^2 v_x}{dt^2} = \omega \frac{d v_y}{dt} = -\omega^2 v_x$$

$$\frac{d^2 v_y}{dt^2} = -\omega \frac{d v_x}{dt} = -\omega^2 v_y$$

$$\frac{d^2 v_x}{dt^2} + \omega^2 v_x = 0 \qquad \frac{d^2 v_y}{dt^2} + \omega^2 v_y = 0$$

Simple Harmonic Oscillator

$$v_x(t) = A \cos \omega t + B \sin \omega t$$

$$v_y(t) = C \cos \omega t + D \sin \omega t$$

Use original equations

$$\frac{dv_x}{dt} = \omega v_y$$

$$-\omega A \sin\omega t + \omega B \cos\omega t = \omega\left(C\cos\omega t + D\sin\omega t\right)$$

$$\Longrightarrow \quad -A = D, \quad B = C$$

$$v_x(t) = A\cos\omega t + B\sin\omega t$$

$$v_y(t) = B\cos\omega t - A\sin\omega t$$

Initial Conditions

$$v_x(0) = 0 \quad \Longrightarrow \quad A = 0$$

$$v_y(0) = v_0 \quad \Longrightarrow \quad B = v_0$$

$$v_x(t) = v_0 \sin\omega t \qquad v_y(t) = v_0\cos\omega t$$

Integrate

$$x(t) = -\frac{v_0}{\omega}\cos\omega t + C_x \qquad y(t) = \frac{v_0}{\omega}\sin\omega t + C_y$$

Initial Conditions

Define $R = \frac{v_0}{\omega}$ radius of orbit

$$x(0) = x_0 = -R + C_x$$

$$C_x = x_0 + R$$

$$y(0) = 0 \implies C_y = 0$$

$$x(t) = -R\cos\omega t + x_0 + R$$

$$y(t) = R\sin\omega t$$

Let $x' = x - x_0 - R$

$$x'(t) = -R\cos\omega t$$

$$y(t) = R\sin\omega t$$

$$x'^2 + y^2 = R^2\cos\omega t + R^2\sin\omega t = R^2$$

Circle of radius R centered at $x_0 + R$

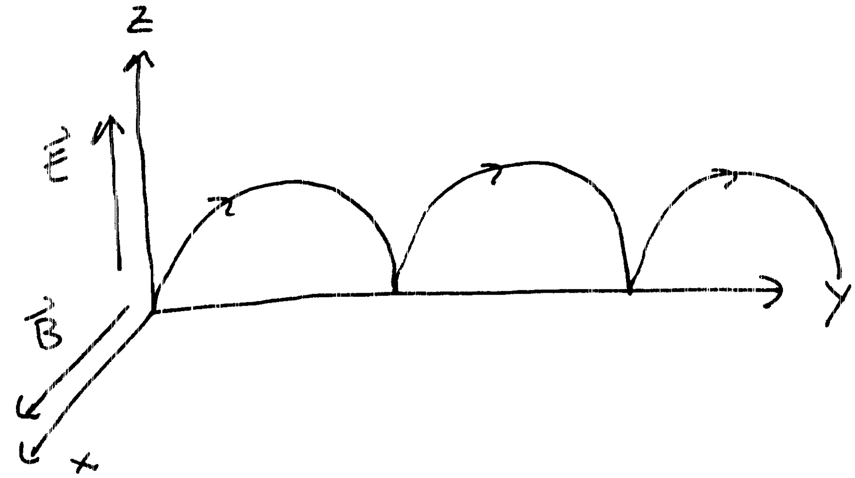
Cirle traversed in clockwise direction.

Now try crossed electric and magnetic fields.

$$\vec{E} = E_0 \hat{z} \qquad \vec{B} = B_0 \hat{x}$$

The forces in the x direction is zero. So if the initial velocity in the x direction is zero, the particle moves in the y-z plane.

The trajectory if released from rest is



$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \qquad v_x = 0$$

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & v_y & v_z \\ B_0 & 0 & 0 \end{vmatrix}$$

$$= + v_z B_0 \hat{y} - v_y B_0 \hat{z}$$

The total force is

$$\vec{F} = q E_o \hat{z} - q v_y B_o \hat{z} + q v_z B_o \hat{y}$$

$$= m \vec{a}$$

$$\vec{a} = \left(\frac{q E_o}{m} - v_y \omega \right) \hat{z} + \omega v_z \hat{y}$$

$$= \omega \left(\frac{E_o}{B_o} - v_y \right) \hat{z} + \omega v_z \hat{y}$$

$$\omega = \frac{qB}{m}$$

Equations of Motion

$$\frac{dv_y}{dt} = a_y = \omega v_z$$

$$\frac{dv_z}{dt} = -\omega v_y + \omega \frac{E_o}{B_o}$$

$$\frac{d^2 v_y}{dt^2} = \omega \frac{dv_z}{dt} = -\omega^2 v_y + \omega^2 \frac{E_o}{B_o}$$

$$\frac{d^2 v_z}{dt^2} = -\omega \frac{dv_y}{dt} = -\omega^2 v_z$$

Solutions

$$v_y = A\sin\omega t + B\cos\omega t + E_0/B_0$$

$$v_z = C\sin\omega t + D\cos\omega t$$

$$\frac{dv_y}{dt} = A\omega\cos\omega t - B\omega\sin\omega t = \omega v_z$$

$$= \omega\left(C\sin\omega t + D\cos\omega t\right)$$

$$\Longrightarrow A = D, \quad C = -B$$

So,

$$v_y = A\sin\omega t + B\cos\omega t + E_0/B_0$$

$$v_z = -B\sin\omega t + A\cos\omega t$$

$$= -B\sin\omega t + A\cos\omega t$$

Integrate

$$y(t) = -\frac{A}{\omega}\cos\omega t + \frac{B}{\omega}\sin\omega t + \frac{E_0}{B_0}t + C_x$$

$$z(t) = \frac{B}{\omega}\cos\omega t - \frac{A}{\omega}\sin\omega t + C_z$$

Initial Conditions (Release $\vec{V}_0 = 0$)

$$V_y = 0 \implies B + \frac{E_0}{B_0} = 0$$

$$B = -\frac{E_0}{B_0}$$

$$V_z = 0 \implies A = 0$$

So,

$$y(t) = -\frac{E_0}{\omega B_0} \sin \omega t + \frac{E_0}{B_0} t + C_y$$

$$z(t) = -\frac{E_0}{\omega B_0} \cos \omega t + C_z$$

$$y(0) = 0 \implies C_y = 0$$

$$z(0) = 0 \implies \frac{-E_0}{\omega B_0} + C_z = 0$$

$$C_z = +\frac{E_0}{\omega B_0}$$

$$y(t) = \frac{E_0}{B_0 \omega}\left(\omega t - \sin \omega t\right)$$

$$z(t) = \frac{E_0}{B_0 \omega}\left(1 - \cos \omega t\right)$$

Eliminate sin, cos and let $R = E_0/B_0 \omega$

$$\left(y(t) - R\omega t\right)^2 + \left(z(t) - R\right)^2 = R^2 \sin^2 \omega t + R^2 \cos^2 \omega t$$

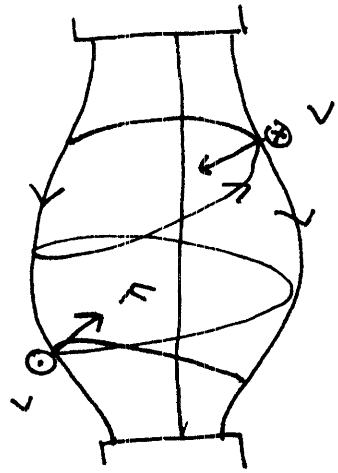
$$= R^2$$

Equation of a circle with center

$$(0, R\omega t, R)$$

Velocity of center $v_c = R\omega = \frac{E_0}{B_0}$

Magnetic Bottle -



Region with field gradient, force pushes particles back toward center.