Magnetic Multipole Expansion

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}\,'}{r''}$$

$$\frac{1}{r''} = \frac{1}{r} \sum_n \left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

as before.

$$\vec{A} = \frac{\mu_0 I}{4\pi} \sum_n \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta') \, d\vec{l}\,'$$

Monopole Term - $n=0$

$$\vec{A}_{mono} = \frac{\mu_0 I}{4\pi r} \int d\vec{l}\,' \qquad P_0 = 1$$

If current path is closed, $\int d\vec{l}\,' = 0$,

so as promised there is no monopole term

Dipole Term $n=1$

$$\vec{A}_{dip} = \frac{\mu_0 I}{4\pi r^2} \int r' \cos\theta' \, d\vec{l}'$$

$P_1 = \cos\theta'$

As before, $\hat{r}\cdot\vec{r}' = r'\cos\theta'$. Note, θ' is the angle between $\vec{r}$ and $\vec{r}'$ in expansion

$$\vec{A}_{dip} = \frac{\mu_0 I}{4\pi r^2} \int (\hat{r}\cdot\vec{r}') \, d\vec{l}'$$

From problem 61, or your additional table of integral identities

$$\oint_C \vec{c}\cdot\vec{r} \, d\vec{l} = \vec{a}\times\vec{c} \qquad (\vec{a} = a\hat{n})$$

area

If $\vec{c} = \hat{r}$
$$\oint_C (\hat{r}\cdot\vec{r}') \, d\vec{l}' = \vec{a}\times\hat{r}$$

where $\vec{a} \equiv \int_S \hat{n} \, da$

$$\oint_C (\hat{r} \cdot \vec{r}\,')\, d\vec{l}\,' = \left(\int \hat{n}\, da \right) \times \hat{r}$$

$$= -\hat{r} \times \int_S \hat{n}\, da$$

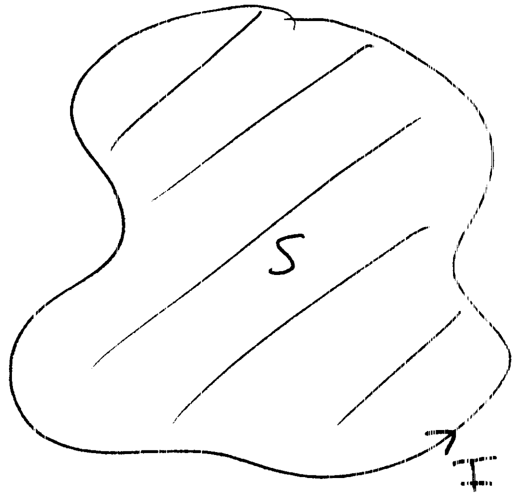
$$\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} \cdot \left(I \int_S \hat{n}\, da \right) \times \hat{r}$$

<u>Magnetic Dipole Moment</u> $\vec{m}$

$$\vec{m} = I \int_S \hat{n}\, da$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

Geometry



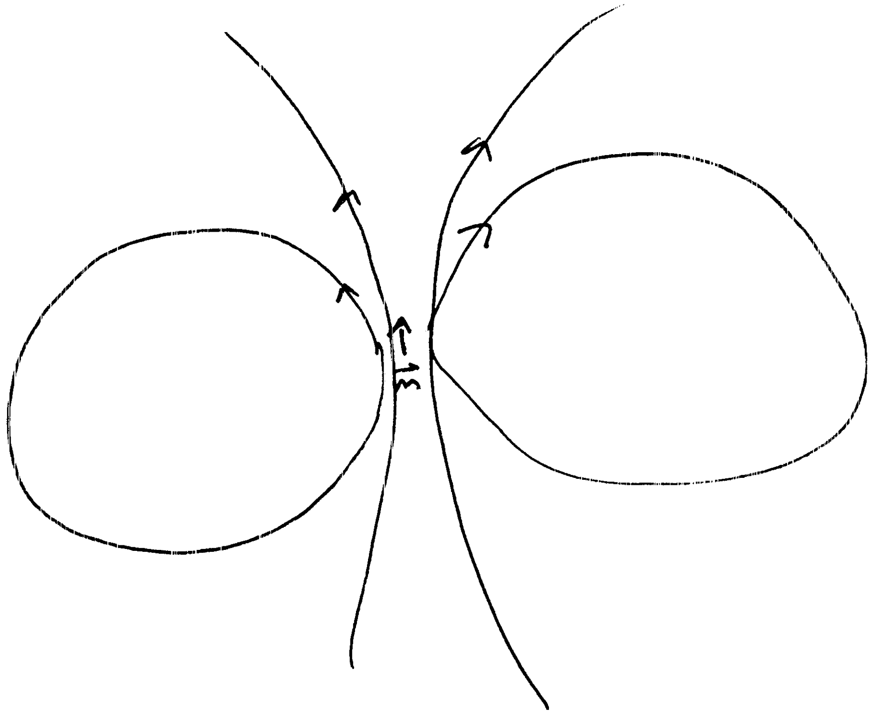
Magnetic Dipole Field

$$\vec{B}_{dip} = \nabla \times A_{dip} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

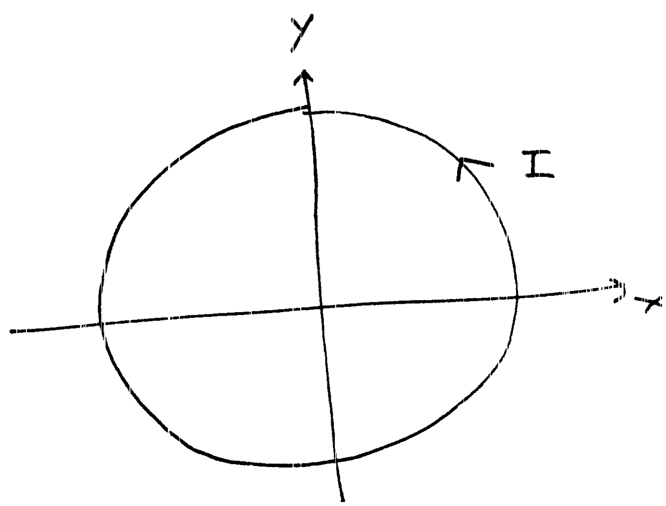
$\Rightarrow$ For dipole $\vec{m} = m\,\hat{y}$ this field becomes

x-axis $\quad \vec{B} = -\dfrac{\mu_0 m}{4\pi x^3}\,\hat{y}$

y-axis $\quad \vec{B} = \dfrac{2\mu_0 m}{4\pi y^3}\,\hat{y}$



Ex Loop of current in x-y plane of radius R



RHR Moment - Curl fingers in the direction of I, thumb points in the direction of moment.

$\Rightarrow$ Come from convention for positive normal.

Sln

$$\vec{m} = I A \hat{z} = \pi R^2 I \hat{z}$$

Field along z axis far from loop

$$\vec{B} = \frac{2 m \mu_0}{4\pi z^3} \hat{z}$$

$$= \frac{2\pi R^2 I \mu_0}{4\pi z^3} \hat{z} = \frac{I R^2 \mu_0}{2 z^3} \hat{z}$$

But we know the exact field.

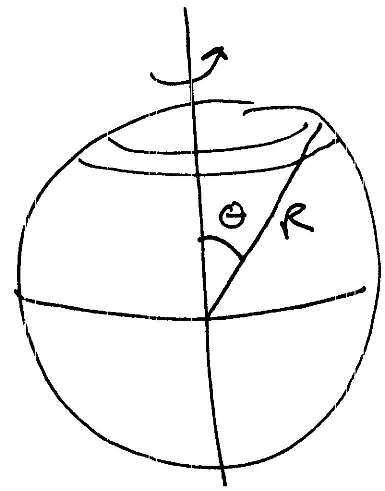
$$B(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$$

$$\cancel{= \frac{\mu_0 I}{2R} \left(1 + \left(\frac{z}{R}\right.\right.}$$

$$B(z) = \frac{\mu_0 I R^2}{2 z^3} \left(1 + \left(\frac{R}{z}\right)^2 \right)^{-3/2}$$

$$= \frac{\mu_0 I R^2}{2 z^3} \quad \text{at } z \to \infty$$

Ex Magnetic Dipole Moment of thin shell of charge σ spinning with angular velocity σ.



Soln $\vec{K} = \sigma\omega R \sin\theta \, \hat{\phi}$ as before.

Slice the surface up into loops of width $R\,d\theta$

The current in each loop is $I = \sigma\omega R^2 \sin\theta \, d\theta$
$= K R \, d\theta$

The magnetic moment of each loop is

$$m = IA = I \pi (R \sin\theta)^2$$

so

$$dm = \pi\sigma\omega R^4 \sin3\theta \, d\theta$$

$$\vec{m} = \hat{z} \int_0^{\pi} \pi\sigma\omega R^4 \sin3\theta \, d\theta$$

$$\vec{m} = \hat{z}\, \pi \sigma \omega R^4 \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3}$$

$$\vec{m} = \frac{4}{3} \pi \sigma \omega R^4 \, \hat{z}$$