Maxwell's Equations

In free space, the complete form of Maxwell's equations are

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \qquad \text{Gauss}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{No Magnetic Monopoles}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The sources $\rho, \vec{J}$ are the total charges and currents. Maxwell equations also apply in matter if any charge densities or currents produced by the materials are included in ρ and $\vec{J}$.

In materials, the charge density can be separated into a bound charge density produced by polarization an other stuff called free charge

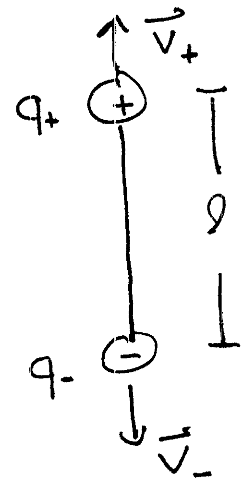
$$\rho = \rho_f + \rho_b$$

Likewise, the current density $\vec{J}$ can be separated into a bound current density resulting from magnetization, a current density $\vec{J}_p$ resulting from <u>changing</u> polarization, and other stuff called free current.

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p$$

<u>Polarization Current</u> $\vec{J}_p$

Consider a simple model of the polarization as a density ρ (dipoles per volume) of stick dipoles



$$q_+ = q$$

$$q_- = -q$$

$$v_+ = \frac{1}{2}\frac{dl}{dt}$$

$$v_- = -\frac{1}{2}\frac{dl}{dt}$$

The polarization is

$$P = \rho q l$$

The time rate of change of polarization is

$$\frac{dP}{dt} = \rho q \frac{dl}{dt}$$

The current produced by the change in polarization is

$$|\vec{J}_p| = \rho q_+ v_+ + \rho q_- v_-$$

$$= \frac{1}{2}\rho q \frac{dl}{dt} + \frac{1}{2}\rho q \frac{dl}{dt}$$

$$= \rho q \frac{dl}{dt} = \frac{dP}{dt}$$

Polarization Current - Current resulting from changes in polarization

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

The total current is then

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p$$

With the division into free and bound currents Maxwell's eqns become

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0}(\rho_f + \rho_b) = \frac{1}{\varepsilon_0}(\rho_f - \nabla \cdot \vec{P})$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{J}_b + J_p \right) + \mu_0 \vec{J}_d$$

$$= \mu_0 \left(\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

<u>Rearranging Gauss</u>

$$\varepsilon_0 \nabla \cdot \vec{E} + \nabla \cdot \vec{P} = \rho_f$$

$$\nabla \cdot \vec{D} = \rho_f$$

where $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ as before

<u>Rearranging Ampere</u>

$$\frac{\nabla \times \vec{B}}{\mu_0} - \nabla \times \vec{M} = \vec{J}_f + \frac{\partial}{\partial t} \left(\varepsilon_0 \vec{E} + \vec{P} \right)$$

with $\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$ this becomes

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Maxwell's Equations in Matter

$$\nabla \cdot \vec{D} = \rho_f \qquad \oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$$

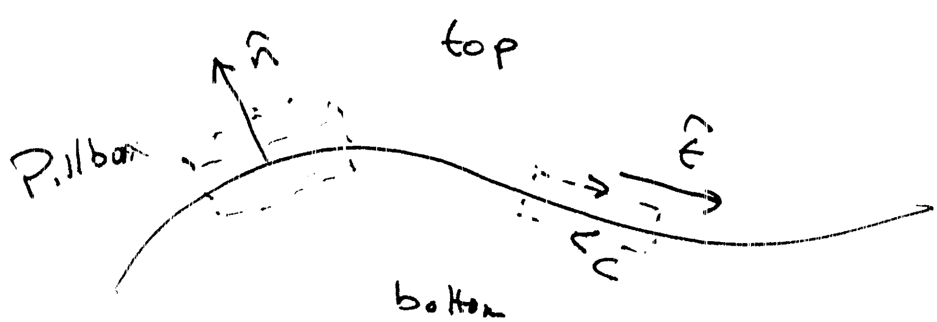
$$\nabla \cdot \vec{B} = 0 \qquad \oint \vec{B} \cdot d\vec{a} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \oint_c \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B} \cdot d\vec{a}$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \qquad \int_c \vec{H} \cdot d\vec{l} = I_{f_{enc}} + \frac{d}{dt}\int_s \vec{D} \cdot d\vec{a}$$

Boundary Conditions

Apply Maxwell to Gaussian Pillbox + Amperian Path at Surface



Gauss

$$\vec{D}_t \cdot \hat{n} - \vec{D}_b \cdot \hat{n} = \sigma_f$$

$$D_t^{\perp} - D_b^{\perp} = \sigma_f$$

No Monopoles

$$\vec{B}_t \cdot \hat{n} - \vec{B}_b \cdot \hat{n} = 0$$

$$B_t^{\perp} = B_b^{\perp}$$

Faraday

$$\vec{E}_t \cdot \hat{t} - \vec{E}_b \cdot \hat{t} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a} = 0$$

$$E_t^{\parallel} = E_b^{\parallel}$$

since flux goes to zero as height of pillbox goes to zero.

Ampere

$$\left| \vec{H}_t \cdot \hat{t} - \hat{H}_b \cdot \hat{t} \right| = \left| I_{f\,enc} \right|$$

to get sign we need to be careful

$$\vec{H}_t^{\parallel} - \vec{H}_b^{\parallel} = \vec{K}_f \times \hat{n}$$

If the material is linear

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}$$

$$\mu_0 \mu_r \vec{H} = \cancel{B}\,\vec{B}$$

The boundary conditions become

$$\varepsilon_t \varepsilon_0 E_t^{\perp} - \varepsilon_b \varepsilon_0 E_b^{\perp} = \sigma_f$$

$$B_t^{\perp} = B_b^{\perp}$$

$$\vec{E}_t^{\parallel} = \vec{E}_b^{\parallel}$$

$$\frac{\vec{B}_t^{\parallel}}{\mu_t \mu_0} - \frac{\vec{B}_b^{\parallel}}{\mu_b \mu_0} = \vec{K}_f \times \hat{n}$$