Vector Fields

The vanishing of the second vecter derivatives

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \quad \text{and} \quad \nabla \times \nabla f = 0$$

will allow a dramatic simplification of our mathematically treatment of electromagnetic fields.

Helmholtz Thm - If a vector field goes to zero at infinity, the field is uniquely determined by its divergence and curl.

$\Rightarrow$ This is good because what Maxwell's eqns give us is the div and curl of $\vec{E}$ and $\vec{B}$.

~~By Helmholtz thm, if the curl of a function is zero then the~~

Curl-Free Fields

If $\nabla \times \vec{F} = 0$, then

- $\int_{\vec{a}\to\vec{b}} \vec{F}\cdot d\vec{l}$ is independent of poth
- $\oint \vec{F}\cdot d\vec{l} = 0$ for any closed loop.
- $\vec{F} = -\nabla V$
- The field is irrotaticnal.

Divergence-less Fields

If $\nabla\cdot\vec{F} = 0$,

- $\vec{F} = \nabla\times\vec{A}$ for some $\vec{A}$
- $\int_S \vec{F}\cdot d\vec{a}$ is independent of surface for any C.
- $\oint \vec{F}\cdot d\vec{a} = 0$ for all closed surfaces.
- The field is solenoidal

Maxwell's Eqns

Gauss' Law

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Gauss' Law for Magnetism (No Magnetic Monopoles)

$$\oint \vec{B} \cdot d\vec{a} = 0$$

Faraday's Law

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a}$$

Ampere's Law

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \int_S \vec{E} \cdot d\vec{a}$$

What do the terms in Maxwell's eqns mean?

Electric Field $\vec{E}$

Magnetic Field $\vec{B}$

<u>Constants</u>

Permittivity of Free Space

$$\varepsilon_0 = 8.85 \times 10^{-12} \, C^2/Nm^2$$

Permeability of Free Space

$$\mu_0 = 4\pi \times 10^{-7} \, \frac{Tm}{A}$$

Note The speed of light, c, is defined to be exactly

$$c = 299,792,458 \, m/s$$

and

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = \frac{1}{(4\pi \times 10^{-7})(299,792,458)^2} \, \frac{C^2}{Nm^2}$$

exactly

Charge Density

Volume Charge Density (ρ) - Charge per unit volume

Surface Charge Density (σ) - Charge per unit area.

Linear Charge Density (λ) - Charge per unit length.

Total Charge in surface S

$$Q_{enc} = \int_V \rho \, d\tau$$

Current Density $\vec{J}$ - Charge per unit area per unit time passing through some surface.

Total Current passing through surface S

$$I_{enc} = \int_S \vec{J} \cdot d\vec{a}$$

Back to Maxwell

Gauss

$$\underbrace{\oint_S \vec{E}\cdot d\vec{a} = \int_V \nabla\cdot\vec{E}\, d\tau}_{\text{divergence thm}} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau$$

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0$$ Gauss' Law

Gauss for Magnetism

$$\oint_S \vec{B}\cdot d\vec{a} = 0 = \int_V (\nabla\cdot\vec{B})\, d\tau = 0$$

for all V

$$\nabla\cdot\vec{B} = 0$$

Faraday

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}$$

$$= \int_S (\nabla\times\vec{E})\cdot d\vec{a}$$ Stokes

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Ampere

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{a}$$

$$= \mu_0 \int_S \vec{J}\cdot d\vec{a} + \mu_0 \varepsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{a}$$

Stokes

$$\oint (\nabla\times\vec{B})\cdot d\vec{a} =$$

$$\nabla\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The magnetic field is solenoidal $\nabla\cdot\vec{B} = 0$ so $\exists$ a function $\vec{A}$ called the vector potential s.t. $\vec{B} = \nabla\times\vec{A}$.

If $\vec{B}$ is constant, the electric field is curl free, $\nabla\times\vec{E} = 0 \Longrightarrow \exists$ a function V s.t. $\vec{E} = -\nabla V$. V is the scalar potential.

If B is changing,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{A}$$

$$= \nabla \times \left(-\frac{\partial \vec{A}}{\partial t} \right)$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Differential Form of Maxwell's Eqns

<u>Gauss</u> $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

<u>No Mag Monopoles</u> $\nabla \cdot \vec{B} = 0$

<u>Faraday</u> $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

<u>Ampere</u> $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$