Maxwell's Eqns for Magnetism

Let's compute $\nabla \cdot \vec{B}$ and $\nabla \times \vec{B}$ from the Biot-Savart Law.

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \int_V \nabla \cdot \left(\frac{\vec{J}(\vec{r}\,') \times \hat{r}''}{r''^2} \right) d\tau'$$

<u>Identity</u>

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \cdot \left(\vec{J} \times \left(\frac{\hat{r}''}{r''^2} \right) \right) = \frac{\hat{r}''}{r''^2} \cdot (\nabla \times \vec{J}) - \vec{J} \cdot \left(\nabla \times \frac{\hat{r}''}{r''^2} \right)$$

$\nabla \times \vec{J}(\vec{r}\,') = 0$ because $\vec{J}$ depends only on $\vec{r}\,'$ coordinates, but derivative taken with respect to un-primed coordinates

$$\nabla \times \left(\frac{\hat{r}''}{r''^2} \right) = 0$$

because this is just the curl of the point charge radial dependence.

so $\nabla \cdot \vec{B} = 0$

No Magnetic Monopoles

- $\nabla \cdot \vec{B} = 0 \quad \longleftrightarrow \quad \int \vec{B} \cdot d\vec{a} = \Phi_m = 0$
- Total magnetic flux out of a closed surface is zero.
- No isolated magnetic charge.
- Magnetic field lines are closed curves.

Examine $\nabla \times \vec{B}$

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int_V \nabla \times \left(\vec{J}(\vec{r}\,') \times \frac{\hat{r}''}{r''^2} \right) d\tau'$$

Identity

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A})$$

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int_V \Bigg[\overset{①}{\left(\frac{\hat{r}''}{r''^2} \cdot \nabla \right) \vec{J}} - \overset{②}{(\vec{J} \cdot \nabla)\left(\frac{\hat{r}''}{r''^2} \right)} + \overset{③}{\vec{J}\left(\nabla \cdot \left(\frac{\hat{r}''}{r''^2} \right) \right)} - \overset{④}{\frac{\hat{r}''}{r''^2} (\nabla \cdot \vec{J})} \Bigg] d\tau'$$

① + ④ = 0 because derivatives taken with respect to unprimed variables, but $\vec{J}(\vec{r}\,')$ depends only on primed variables.

From before,

$$\nabla \cdot \left(\frac{\hat{r}''}{r''^2} \right) = 4\pi \delta(\vec{r}'')$$

so 3 $= \vec{J}(\vec{r}')\, 4\pi\, \delta(\vec{r}'')$

Now work on piece 2

$$\int_V (\vec{J} \cdot \nabla) \left(\frac{\hat{r}''}{r''^2} \right) d\tau'$$

In r'', $\hat{r}''$ the variables x, x', y', y, z, z' only appear in combinations $x-x'$, $y-y'$, $z-z'$

So the effect of the transformation $x \to x'$ is to change signs.

$$(\vec{J} \cdot \nabla) \left(\frac{\hat{r}''}{r''^2} \right) = -(\vec{J} \cdot \nabla') \left(\frac{\hat{r}''}{r''^2} \right)$$

More identities,

$$\nabla \cdot (f \vec{A}) = f(\nabla \cdot \vec{A}) + (\vec{A} \cdot \nabla) f$$

$$(\vec{A} \cdot \nabla) f = \nabla \cdot (f \vec{A}) - f(\nabla \cdot \vec{A})$$

This is relatively complicated, let's look at the x-component. The other components will be similar.

$$\left(\vec{J}' \cdot \nabla'\right)\left(\frac{x-x'}{r''^3}\right) = \nabla' \cdot \left(\frac{x-x'}{r''^3}\,\vec{J}(\vec{r}\,')\right)$$

$$- \frac{x-x'}{r''^3}\left(\nabla' \cdot \vec{J}(\vec{r}\,')\right)$$

The second term is zero for magnetostatics, $\nabla' \vec{J} = 0$.

Integrate the remaining part of 2

$$\int_V \nabla' \cdot \left(\frac{x-x'}{r''^3}\, J(\vec{r}\,')\right) d\tau'$$

$$= \int_S \frac{x-x'}{r''^3}\, J(\vec{r}\,')\, d\tau'$$

$\Rightarrow 0$ if we ~~ensure~~ ensure S encloses all charge, and so no current passes through the surface.

So all that is left is 3

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int_V \vec{J}(\vec{r}\,') \, 4\pi \delta(\vec{r}\,'') \, d\tau'$$

$$= \mu_0 \vec{J}(\vec{r})$$

Ampere's Law (magnetostatics)

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Convert to integral form

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$$

$$d\vec{a} = \hat{n} \, da$$

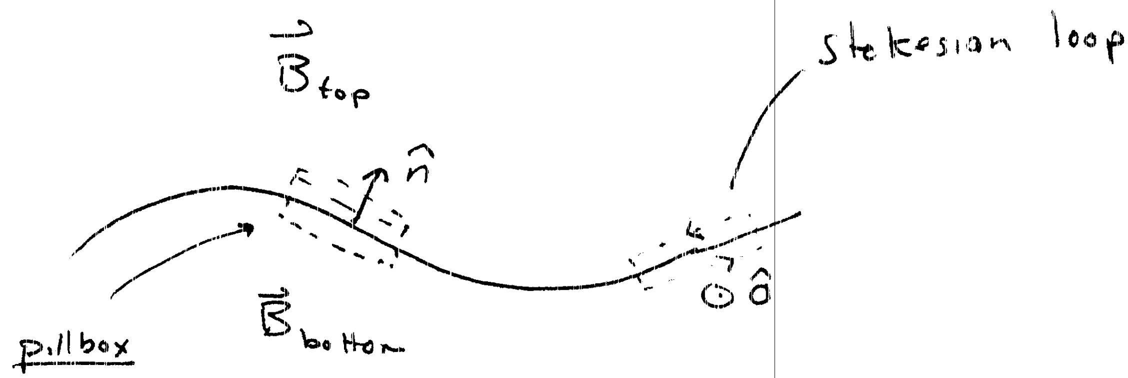
Stoke's Thm

$$\int_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Ampere's Law - Integral Form

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Magnetostatic Boundary Conditions



Apply $\nabla \cdot \vec{B}$ to pillbox

$$\Phi_m = \vec{B}_{top} \cdot \hat{n} A - \vec{B}_{bottom} \cdot \hat{n} A = 0$$

$$B_{top}^{\perp} = B_{bottom}^{\perp}$$

Apply $\nabla \times \vec{B} = \mu_0 I_{enc}$ to Stokesian Loop

$$\vec{B}_{top} \cdot \hat{t}\, l - \vec{B}_{bottom} \cdot \hat{t}\, l = \mu_0 \vec{K} \cdot \hat{a}$$

where $\vec{K}$ is the surface current and $\hat{a}$ is the surface normal to the curve bounded by the stokesian loop.

In vector form these can be combined as

$$\vec{B}_{top} - \vec{B}_{bottom} = \mu_0 (\vec{K} \times \hat{n})$$