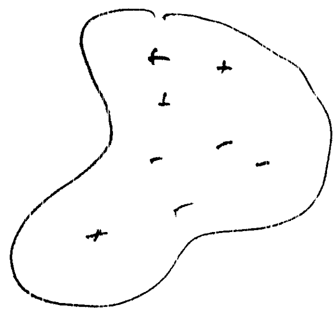
# Multipole Expansions

We wish to develop a consistent way to investigate the potential (and field) of a complicated object like

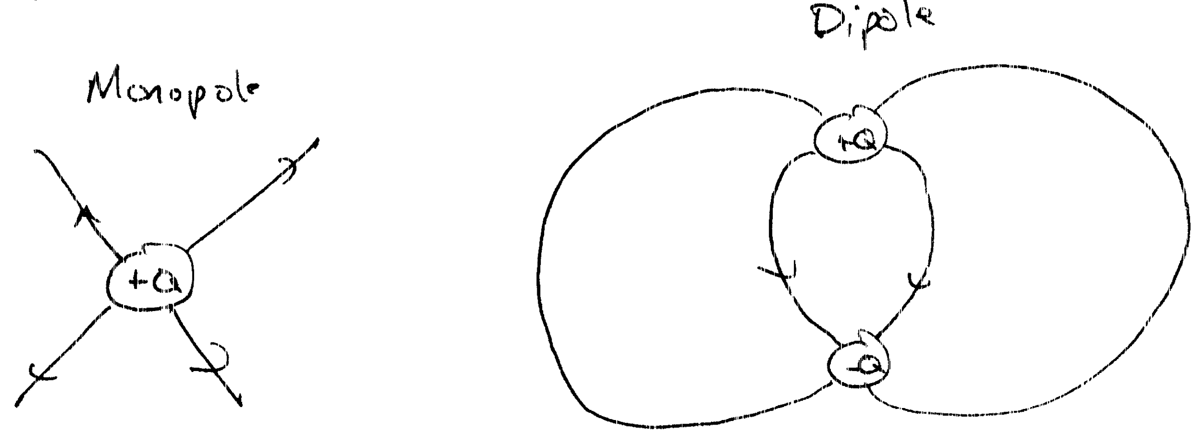


We already have some methods, if $Q_{total} > 0$. The field far from the object will be

$$\vec{E} = \frac{Q_{Total}}{4\pi\epsilon_0 r^2}\hat{r}$$

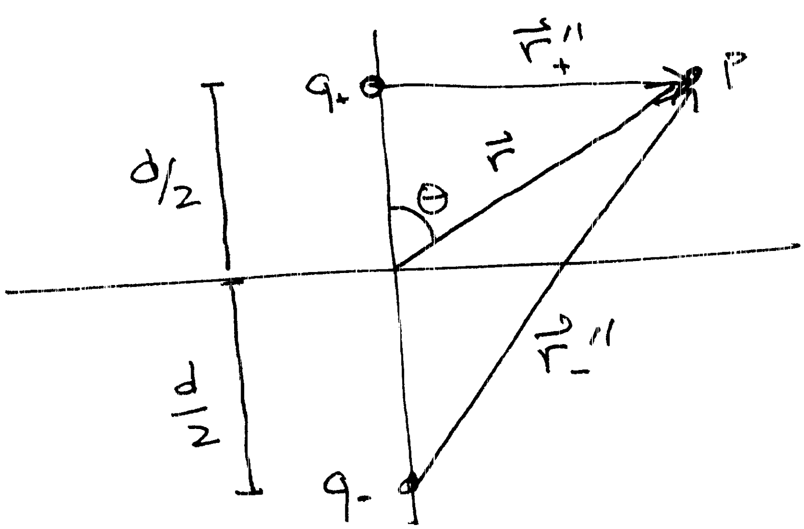
To improve the approximation far from the charge we could investigate the dipole component of the field; the above will be called the monopole field.

We can build a simple dipole by separating equal, but opposite charges




The potential of the simple dipole is

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+''} - \frac{1}{r_-''}\right)$$



This is re-written using the law of cosines and expanded under the assumption that d/r is small to yield

$$V(\vec{r}) = \frac{qd}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2}$$

Define - Dipole Moment Vector ($\vec{p}$) That points from $-q$ to $+q$ and has magnitude $|\vec{p}| = qd$

In terms of $\vec{p}$, $qd\cos\theta = \vec{p}\cdot\hat{r}$

$$V(\vec{r}) = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$$

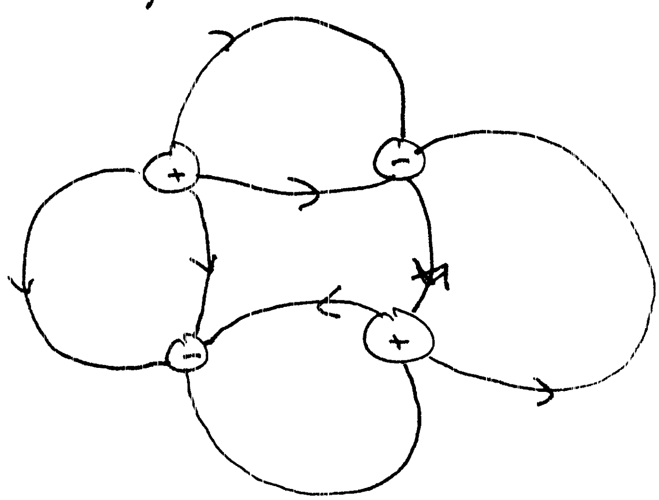
which you are all familiar from homework 1.

$$\vec{E} = -\nabla V = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

(dipole in z direction)

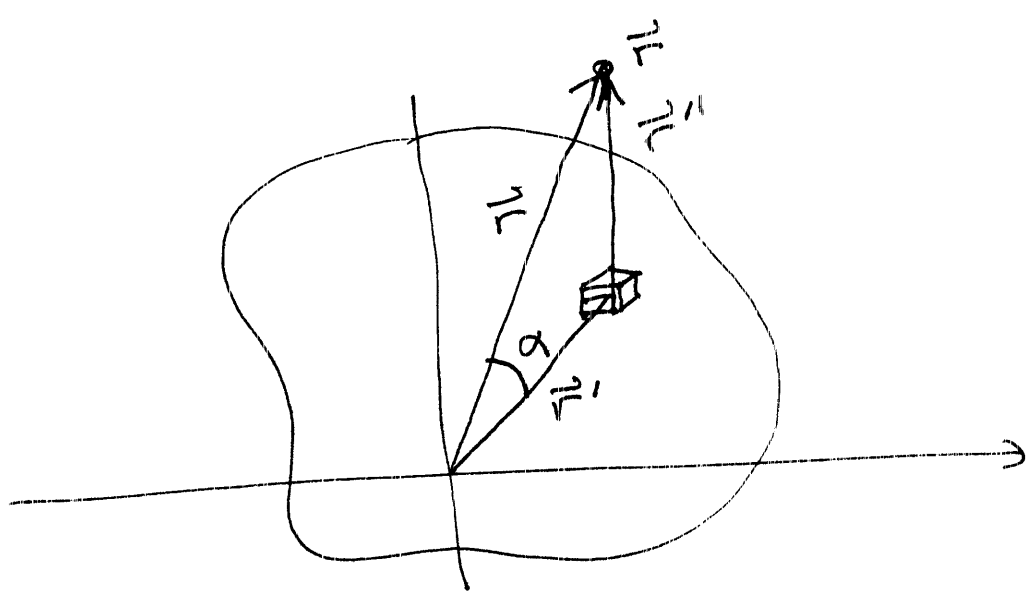
$$= \frac{1}{4\pi\varepsilon_0 r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)$$

We can continue, consider an electric quadrapole



Let's attack the problem in general, for any charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}\,')\, d\tau'}{r''}$$




$$r''^2 = r^2 + r'^2 - 2rr'\cos\alpha \quad \text{(Law of Cosines)}$$

$$r''^2 = r^2 \underbrace{\left[ 1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha \right]}_{s}$$

where if we are a sufficient distance from the object, $s$ is small.

$$r''^2 = r^2(1+s)$$

$$r'' = r\sqrt{1+s}$$

$$\frac{1}{r''} = \frac{1}{r}(1+s)^{-1/2}$$

Binomial Expansion

$$(1+x)^n = 1 + nx + \cdots$$

$$\frac{1}{r''} = \frac{1}{r}\left(1 - \frac{1}{2}s + \frac{3}{8}s^2 - \frac{5}{16}s^3 + \cdots\right)$$

$$s = \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha$$

$$= \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$$

$$\frac{1}{r''} = \frac{1}{r}\left(1 - \frac{r'}{2r}\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2\right.$$

$$\cdots\cdot$$

$$= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2 \frac{(3\cos^2\alpha - 1)}{2} + \cdots\right.$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$V(\vec{r}) = \int \frac{\rho(\vec{r}\,')\,d\tau'}{4\pi\epsilon_0 r''}$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int \rho(\vec{r}\,')\, r'^n P_n(\cos\alpha)\, d\tau$$

Note, $\alpha$ is not constant

Look at first two terms

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0 r} \int \rho(\vec{r}\,')\, d\tau'$$

$$+ \frac{1}{4\pi\epsilon_0 r^2} \int r' \cos\alpha\, \rho(\vec{r}\,')\, d\tau'$$

$$+ \cdots$$

## Monopole Moment

$$Q_T = \text{total charge} = \int \rho(\vec{r}\,')\, d\tau'$$

- Note, field measure from origin, not center of charge

## Dipole Term

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^2} \int r' \cos\alpha\, \rho(\vec{r}\,')\, d\tau'$$

Since $\alpha$ is the angle between $\vec{r}$ and $\vec{r}\,'$

$$\vec{r}\,' \cdot \hat{r} = r' \cos\alpha$$

$$V_{dip}(\vec{r}) = \frac{\hat{r}\,\cdot}{4\pi\epsilon_0 r^2} \int \vec{r}\,' \rho(\vec{r}\,')\, d\tau'$$

## Dipole Moment Vector

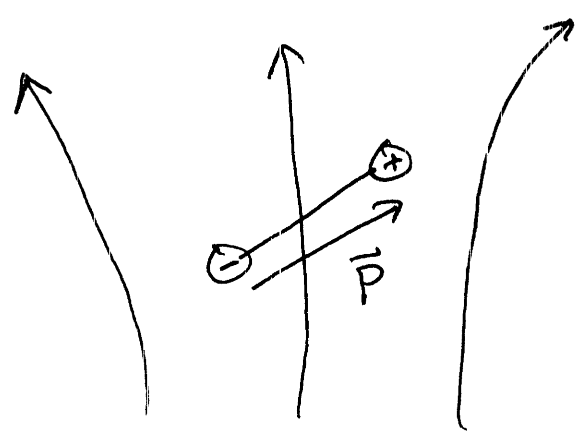
$$\vec{p} = \int \vec{r}\,' \rho(\vec{r}\,')\, d\tau'$$

## Notes

- Dipole moment of point charge not at origin is non-zero, in general dipole moment depends on choice of origin.

- If $Q_T = 0$, dipole moment is independent of origin.

## Dipole Mechanics

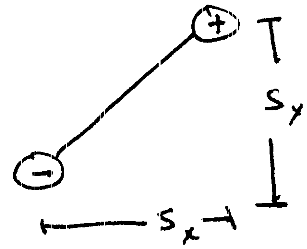



Torque $\quad \vec{\tau} = \vec{p} \times \vec{E}$

Potential Energy

$$U = \int_{\pi/2}^{\theta} \tau \, d\theta = -\vec{p} \cdot \vec{E}$$

## Force on Dipole




$$F_y = Q \Delta E_y = Q \frac{\partial E_y}{\partial y} s_y = p_y \frac{\partial E}{\partial y}$$

But field also changes in x, y directions

$$F_y = p_x \frac{\partial E_y}{\partial x} + p_y \frac{\partial E_y}{\partial y} + p_z \frac{\partial E_y}{\partial z}$$

$$\vec{F} = (\vec{p} \cdot \nabla) \vec{E}$$

If static, $\nabla \times \vec{E} = 0$

$$\vec{F} = \nabla (\vec{p} \cdot \vec{E})$$

Ex Multipole expansion of point charge at

$$\vec{r}_A = (0,0,0) \Rightarrow \rho(\vec{r}) = Q\delta(\vec{r}-\vec{r}_A)$$

Monopole

$$Q_m = \int \rho(\vec{r}\,')\,d\tau' = Q$$

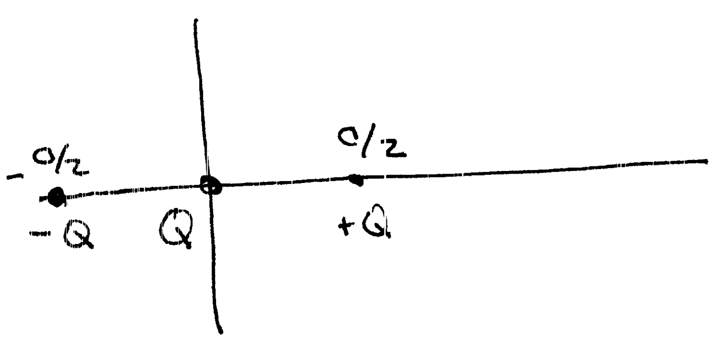
Dipole

$$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'$$

$$= \int \vec{r}\,' Q\,\delta(\vec{r}\,'-\vec{r}_A)\,d\tau'$$

$$= Q\vec{r}_A$$

+ Additional terms



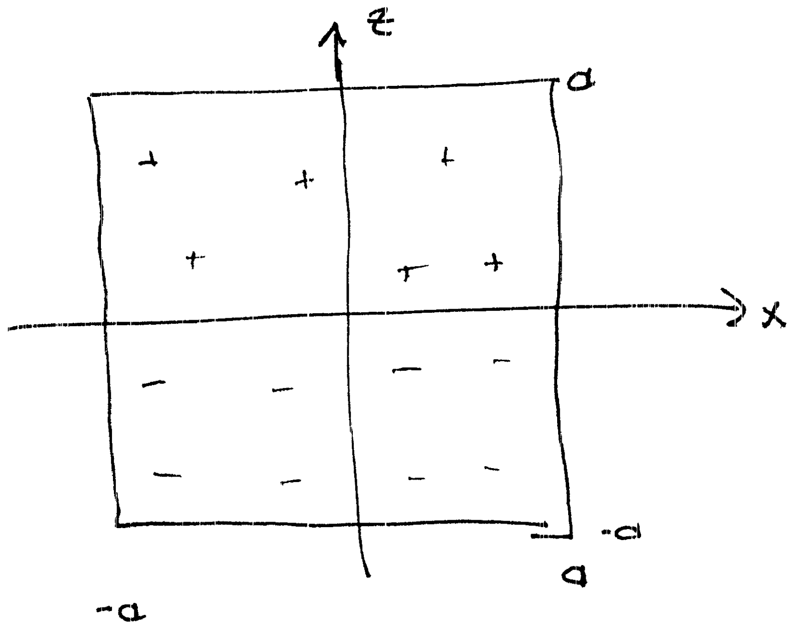
Notes

(1) Not a pure monopole field.

(2) Dipole moment depends on origin

Ex Dipole moment of square region in x-z plane s.t.

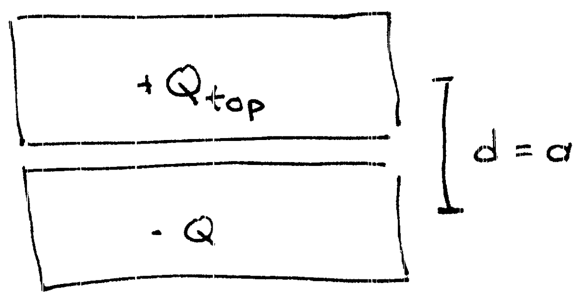
$$\sigma(x) = \begin{cases} +\sigma_0 & z > 0 \\ -\sigma_0 & z < 0 \end{cases}$$




Dipole Moment

$$\vec{P} = \int \vec{r}\,' \sigma(\vec{r}\,')\, da'$$

Does this make sense?




$$Q_{top} = (a)(2a)\sigma = 2a^2\sigma$$

$$|\vec{p}| = d\,Q_{top}$$

Suppose now we wish the field at a point, same distance from the dipole, say $\vec{r}_A = (10a, 0, 10a)$

$$\vec{E}_{dip} = \frac{p}{4\pi\varepsilon_0 r^2}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$\theta$ - Angle from $z$ axis (if dipole direction)

$\theta = 45^\circ \quad r = 10\sqrt{2}\,a$

$$p_x = \int_{-a}^{a}\int_{-a}^{a} x'\sigma(\vec{r}')\,dx'\,dz'$$

$$= \int_{-a}^{0} dz'\,(-\sigma_0)\underbrace{\int_{-a}^{a} x'\,dx'}_{0} + \int_{0}^{a} dz'\,\sigma_0 \underbrace{\int_{-a}^{a} x'\,dx'}_{0}$$

$p_x = 0$ , $p_y = 0$ because $y' = 0$ on plane

$$p_z = \int_{-a}^{a} dz' \int_{-a}^{a} dx'\,\sigma(\vec{r}')\,z'$$

$$= \int_{-a}^{0} z'\,dz'\,(-\sigma_0)\underbrace{\int_{-a}^{a} dx'}_{2a} + \int_{0}^{a} z'\,dz'\,\sigma_0 \underbrace{\int_{-a}^{a} dx'}_{2a}$$

$$= -2a\sigma_0 \int_{-a}^{0} z'\,dz' + 2a\sigma_0 \int_{0}^{a} z'\,dz'$$

$$= 2a\sigma_0 \cdot \frac{a^2}{2} + 2a\sigma_0 \frac{a^2}{2}$$

$$= 2a^3\sigma_0$$