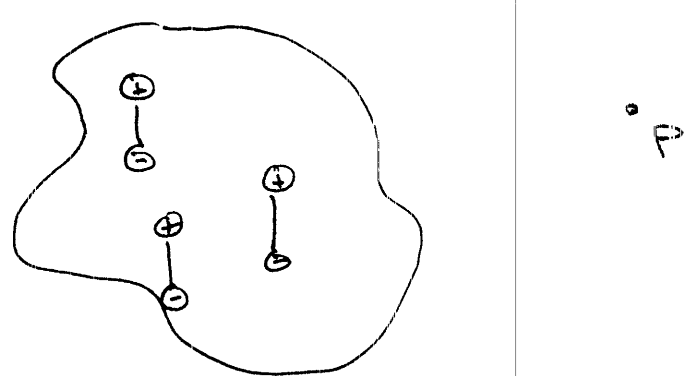
Polarization

We can imagine building a material out of our stick dipoles.



Each dipole produces a potential $V_{dip} = \frac{k\vec{p}\cdot\hat{r}}{r^3}$

The potential of the system at a point P is the sum of the dipole potentials

$$V(\vec{r}_p) = \sum \frac{k\vec{p}\cdot\hat{r}''}{r''^3}$$

If the dipoles are small and numerous, we could describe the system using a dipole moment per unit volume.

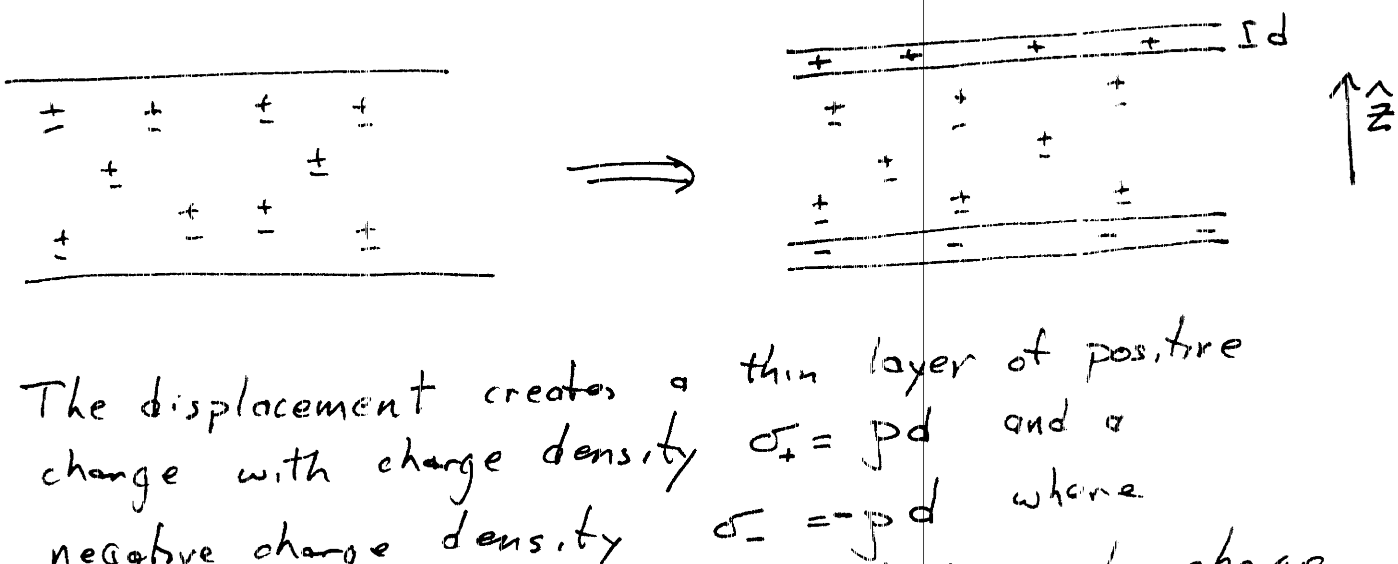
Dfn <u>Polarization</u> ($\vec{P}$) - Dipole moment per unit volume.

The potential at point P then becomes

$$V(\vec{r}_p) = \int_V \frac{\vec{P}\cdot\vec{r}''}{r''^3}\, d\tau'$$

Let's consider some simple models

We could build a dipole moment per unit volume by taking a neutral object made of + and − charge and displacing the positive charge density upward d.



The displacement creates a thin layer of positive charge with charge density $\sigma_+ = \rho d$ and a negative charge density $\sigma_- = -\rho d$ where ρ is the volume charge density of the positive charge.

The dipole moment per unit volume produced by the displacement is

$$\vec{P} = \rho d \hat{z}$$

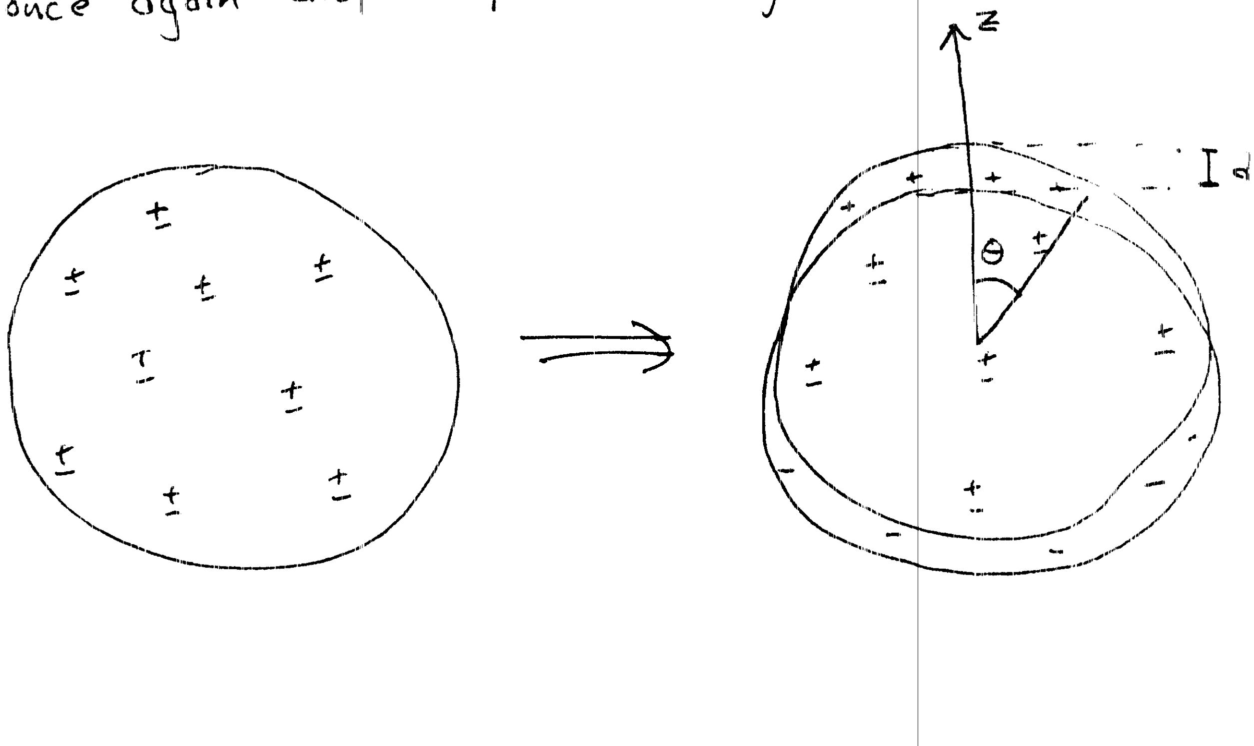
therefore the charge density at the surface

$$\sigma = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{z}$$

The electric field is the field of equal, but opposite planes of charge.

$$\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{z} = -\frac{\vec{P}}{\varepsilon_0}$$

Another example Sphere with uniform polarization; once again displace positive charge density ρ.



Once again, the polarization produces a surface charge density but no volume charge density.

The thickness of the surface layer is $d\cos\theta$ and the surface charge density is $\rho d\cos\theta$. The polarization is once again, $\vec{P} = \rho d$. The total dipole moment of the system is

$$\vec{p} = \vec{P}V = \frac{4}{3}\pi R^3 \rho d\,\hat{z}$$

$$= Qd\,\hat{z} \quad \text{(no suprise)}$$

where Q is the total charge of the system.

We can the compute the long range potential simply by using a dipole potential

$$V(\vec{r}_p) = \frac{k\,\vec{p}\cdot\vec{r}}{r^3}$$

The above is an approxmation; we can also compute the potential exactly.

The only physical effect of polarization for electrostatics is to produce a net charge density $\sigma(\theta) = \rho d \cos\theta$. Once we know this, we can forget polarization exists.

So we need to find the potential of a system with a spherical surface charge $\sigma(\theta) = \rho d \cos\theta = \sigma_0 P_1(\cos\theta)$ where $\sigma_0 = \rho d$.

We know how to do this and have already done it for many similar charge densities, but a test is coming up.

<u>Inside</u> $r < R$

$$V_i = \sum A_n r^n P_n(\cos\theta)$$

<u>Outside</u>

$$V_o = \sum B_n r^{-(n+1)} P_n(\cos\theta)$$

The potential is always continuous

$$V_i(R) = \sum_n A_n R^n P_n(\cos\theta) = \sum_n B_n R^{-(n+1)} P_n(\cos\theta)$$

By orthogonality, those must be equal term by term.

$$A_n R^n = B_n R^{-(n+1)}$$

$$B_n = A_n R^{2n+1}$$

Electrostatic Boundary Condition

$$\left.\frac{\partial V_o}{\partial r}\right|_R - \left.\frac{\partial V_i}{\partial r}\right|_R = \frac{-\sigma}{\varepsilon_0}$$

$$\sum -B_n (n+1) r^{-(n+1)} P_n(\cos\theta)\Big|_R$$
$$- \sum A_n n r^{(n-1)} P_n(\cos\theta)\Big|_R = \frac{-\sigma}{\varepsilon_0}$$

$$\sum -(A_n R^{2n+1})(n+1) R^{-(n+2)} P_n \cos\theta$$
$$- A_n n R^{(n-1)} P_n(\cos\theta) = \frac{-\sigma}{\varepsilon_0}$$

$$\sum -A_n (2n+1) R^{n-1} P_n(\cos\theta) = \frac{-\sigma}{\varepsilon_0}$$

$$\frac{\sigma_0}{\varepsilon_0} P_1(\cos\theta) = \sum A_n (2n+1) R^{n-1} P_n(\cos\theta)$$

By orthogonality again, only the $n=1$ term is non-zero.

$$\frac{\sigma_0}{\varepsilon_0} = A_n 3$$

$$A_n = \frac{\sigma_0}{3\varepsilon_0} = \frac{p d}{3\varepsilon_0} = \frac{|\vec{P}|}{3\varepsilon_0}$$

<u>Inside the Polarized Object</u>

$$V(r,\theta) = A_n r\cos\theta = A_n z$$

So the field inside is

$$\vec{E}_i = -\nabla V = -A_n \hat{z} = -\frac{|\vec{P}|}{3\varepsilon_0}\hat{z}$$

So inside the field is uniform!

Outside the object

$$V_o(r,\theta) = \frac{B_1}{r^2} P_1(\cos\theta)$$

$$= \frac{A_1 R^3}{r^2} \cos\theta$$

$$= \frac{|\vec{P}|\, R^3 \cos\theta}{3\varepsilon_0}$$

Let $\vec{P} = P_0 \hat{z}$. The total dipole moment is

$$\vec{p} = \vec{P} V = \frac{4}{3}\pi R^3 P_0 \hat{z}$$

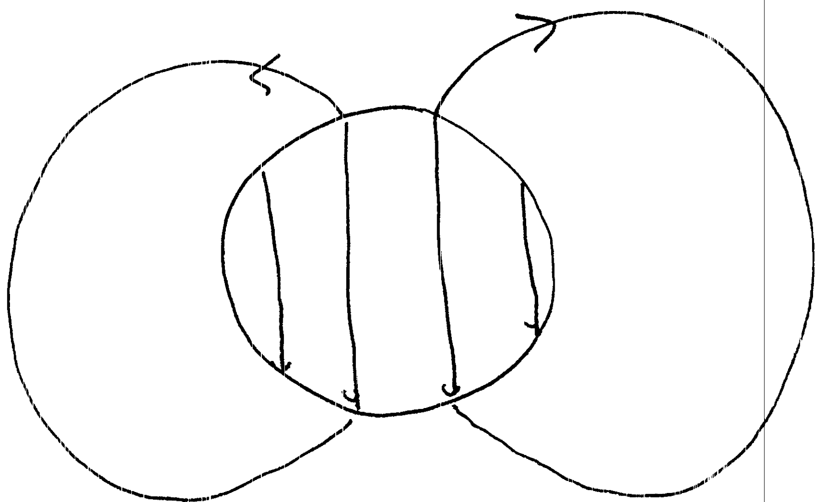
$$V_o(r,\theta) = \frac{|\vec{p}|\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\vec{p}\cdot\hat{r} = |\vec{p}|\cos\theta$$

$$= \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

The potential of a pure dipole.

So inside the field is uniform, outside the field is dipolar.

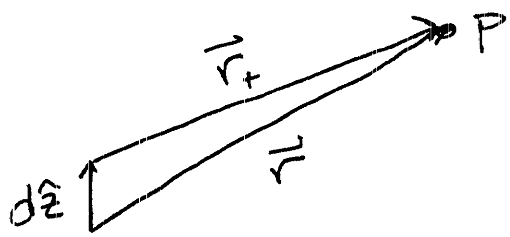


We could attack this in an alternate method. The field of a uniform sphere with charge density ρ is

$$\vec{E} = \begin{cases} \dfrac{\rho \vec{r}}{3\varepsilon_0} & r < R \\[2ex] \dfrac{\frac{4}{3}\pi R^3 \rho}{4\pi\varepsilon_0 r^2}\,\hat{r} & r > R \end{cases}$$

The field inside is the sum of the field of the positive charge density and the negative charge ~~density~~. density.

Let $\vec{r}$ be the displacement vecter from the center of the $-\rho$ sphere and the field point. Let $\vec{r}_+$ be the displacement from the center of the negative sphere to the field point. The positive sphere was displaced a distance $d\hat{z}$



$$d\hat{z} + \vec{r}_+ = \vec{r}$$

The field at P is

$$\vec{E}_P = \frac{-\rho\vec{r}}{3\varepsilon_0} + \frac{\rho\vec{r}_+}{3\varepsilon_0}$$

$$= \frac{-\rho\vec{r}}{3\varepsilon_0} + \frac{\rho}{3\varepsilon_0}\left(\vec{r} - d\hat{z}\right)$$

$$= \frac{-\rho d}{3\varepsilon_0}\hat{z} = -\frac{\vec{P}}{3\varepsilon_0}$$

Check that the dipole moment vector is

$$\vec{p} = \vec{P} V = \frac{4}{3}\pi R^3 \vec{P}$$

for a charge density $\sigma(\theta) = P\cos\theta = \rho d \cos\theta$

The dipole moment is defined as

$$\vec{p} = \int \vec{r}\,'\sigma(\theta')\,da' \qquad da' = (R\,d\theta')(R\sin\theta'\,d\phi')$$

$$p_z = \int z'\underbrace{(P\cos\theta')}_{\sigma}\underbrace{(R\,d\theta')(R\sin\theta'\,d\phi')}_{da}$$

$$z' = R\cos\theta'$$

$$p_z = \int P R^3 \cos^2\theta' \sin\theta'\,d\phi'\,d\theta'$$

$$= P R^3 \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \underbrace{\int_0^{\pi} \cos^2\theta' \sin\theta'\,d\theta'}_{2/3}$$

$$p_z = \frac{4}{3}\pi R^3 P \quad \checkmark$$

The p_x and p_y components are evidently zero.

Now work in general

For any object with dipole moment per unit volume $\vec{P}$

$$V_{dip} = \frac{\hat{r}'' \cdot \vec{p}}{4\pi\varepsilon_0 r''^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\hat{r}'' \cdot \vec{P}(\vec{r}')}{r''^2} d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0} \int_V \left(\frac{\hat{r}''}{r''^2}\right) \cdot \vec{P}(\vec{r}') \, d\tau'$$

We showed earlier

$$\nabla' \left(\frac{1}{r''}\right) = \frac{\hat{r}''}{r''^2}$$

note $\nabla' = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$

So

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \vec{P}(\vec{r}\,') \cdot \nabla'\left(\frac{1}{r''}\right) d\tau'$$

<u>From front cover</u>

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla f$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \nabla' \cdot \left(\frac{\vec{P}(\vec{r}\,')}{r''}\right) d\tau'$$

$$- \frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{r''} \left(\nabla' \cdot \vec{P}(\vec{r}\,')\right) d\tau'$$

<u>Divergence Thm</u>

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oint_S \frac{1}{r''} \vec{P}(\vec{r}\,') \cdot d\vec{a}\,'$$

$$- \frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{r''} \nabla' \cdot \vec{P}(\vec{r}\,')\, d\tau'$$

The first term looks like the potential of a surface charge density, the second a volume charge density.

Dfn - Bound Charge Densities - The real net charge densities resulting from a polarization $\vec{P}$

Bound Surface Charge $\sigma_b = \vec{P} \cdot \hat{n}$

Bound Volume Charge $\rho_b = -\nabla \cdot \vec{P}$

$$V = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma_b \, da'}{r''} + \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_b \, d\tau'}{r''}$$

Check against our test cases

Uniform polarized slab

$$\sigma_b = \vec{P} \cdot \hat{z} = \rho d \checkmark$$

Uniform polarized sphere

$$\sigma_b = \vec{P} \cdot \hat{r} = P \cos\theta = \rho d \cos\theta \checkmark$$

For both cases, $\nabla \cdot \vec{P} = 0$.