Another spherical potential example.

Suppose a potential $V = V_0 \cos\theta + V_1 \cos 2\theta$ is established on the surface of a sphere of radius a.

Compute potential inside surface.

Sln Since inside, throw away $r^{-(n+1)}$ terms

$$V(r,\theta) = \sum A_n P_n(\cos\theta)\, r^n$$

$$V(a,\theta) = V_0 \cos\theta + V_1 \cos 2\theta = \sum A_n a^n r^n$$

Fourier's Trick   Multiple by $P_m(\cos\theta)$

$$\frac{2 A_m a^m}{2m+1} = \int_{-1}^{1} P_m(\cos\theta)\left(V_0 \cos\theta + V_1 \cos 2\theta\right) d\cos\theta$$

What is $P_m(\cos\theta)$?

Look it up in math handbook

$$P_0(\cos\theta) = 1 \qquad P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right)$$

Substitute and integrate

Method II Think like vectors

$$\cos^2\theta = \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}$$

$$= \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)$$

Trig identity

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$V = V_0\cos\theta + 2V_1\cos^2\theta - V_1$$

$$= V_0 P_1(\cos\theta) + 2V_1\left(\frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)\right) - V_1 P_0(\cos\theta)$$

$$= \underbrace{-\frac{1}{3}V_1}_{A_0 a^0} P_0(\cos\theta) + \underbrace{V_0}_{A_1 a} P_1(\cos\theta) + \underbrace{\frac{4}{3}V_1}_{A_2 a^2} P_2(\cos\theta)$$

$$A_0 = -\frac{1}{3}V_1 \qquad A_1 = \frac{V_0}{a} \qquad A_2 = \frac{4V_1}{3a^2}$$

$$V(r,\theta) = -\frac{1}{3}V_1 P_0(\cos\theta) + \frac{V_0}{a} r P_1(\cos\theta) + \frac{4V_1}{3a^2} r^2 P_2(\cos\theta)$$

$$= -\frac{1}{3}V_1 + \frac{V_0}{a} r\cos\theta + \frac{2V_1}{3a^2}\left(3\cos^2\theta - 1\right)$$