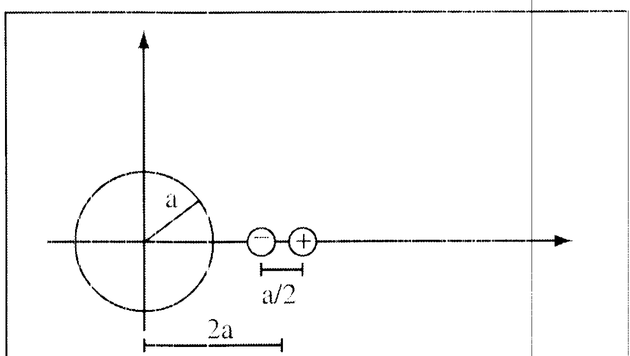
Electricity and Magnetism - Test 1 - Spring 2010

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

Problem 1.1 A spherical object with radius a has a potential at its surface that has value V_0 for a small patch with $0 < \theta < \pi/8$ at its north pole. The potential of the rest of the object is 0. Compute first two non-zero terms of the potential inside the sphere.

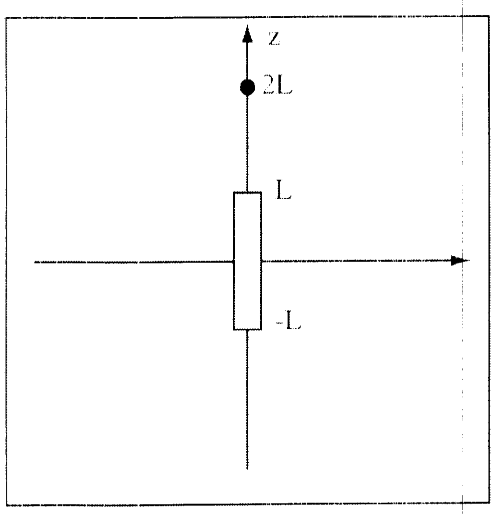
Problem 1.2 A dipole is place outside of a grounded conducting sphere with radius a with its dipole moment pointing in a direction normal to the sphere, as drawn. The charge on the two ends of the dipole are $\pm q$. The center of the dipole is a distance $2a$ from the center of the sphere. The distance between the two charges of the dipole is $a/2$. Compute the force the sphere exerts on the positive charge in the dipole. (I initially wanted the force on the dipole but it was too annoying.)



Problem 1.3 An infinite cylinder of radius a contains a uniform volume charge density ρ. Compute the potential difference between a point on the axis and a point on the outside surface.

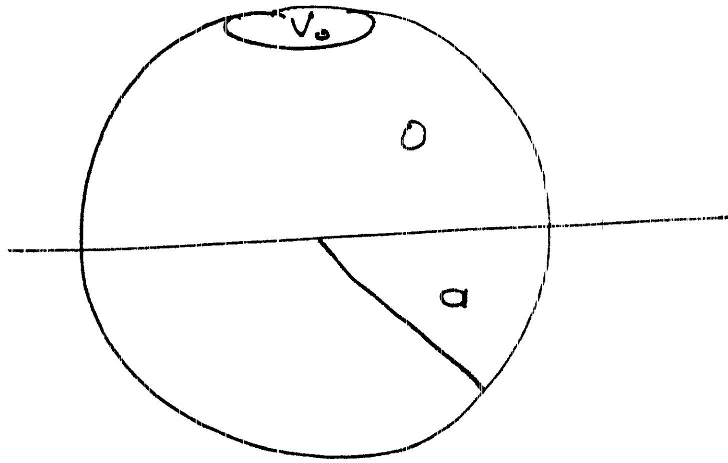
Problem 1.4 A infinite conducting cylinder of radius a has a surface charge density $\sigma(\phi) = \sigma_0(\sin^2(\phi) - \frac{1}{2})$. Compute the potential outside the cylinder.

Problem 1.5 A finite linear charge occupies the region between $-L$ and $+L$ along the z axis and has uniform linear charge density λ. Calculate the electric field at a point a distance $2L$ along the z axis.



Problem 1.6 The electric potential in some region of space is $V = V_0 x^2 + V_0 y^2$. Compute the electric field in cylindrical coordinates. What is the charge density in the region containing the field?

1.1



$$\frac{\pi}{8} \cdot \frac{360}{2\pi} = 22.5^\circ$$

The solution to Laplace's equation in spherical coordinates, discarding solutions that are infinite at the origin.

$$V^i = \sum A_n s^n P_n(\cos\theta) \qquad + 10$$

Apply the boundary condition

$$V^i(a,\theta) = \begin{cases} V_0 & 0 < \theta < \pi/8 \\ 0 & \pi/8 < \theta < \pi \end{cases}$$

$$V^i(a,\theta) = \sum A_n a^n P_n(\cos\theta)$$

Multiply by $P_m(\cos\theta)$

$$I \equiv \int_{-1}^{1} V^i(a,\theta)\, P_m(\cos\theta)\, d(\cos\theta)$$

$$= \sum A_n a^n \underbrace{\int_{-1}^{1} P_n(\cos\theta) P_m(\cos\theta)\, d(\cos\theta)}_{\frac{2}{2m+1}\delta_{nm}}$$

$$= \frac{2 A_m a^m}{2m+1} \quad + 5$$

<u>Work on I</u>

$$I = -\int_{\pi}^{0} V^i(a,\theta)\, P_m(\cos\theta)\, (d\theta) \sin\theta$$

$$= \int_0^{\pi} V^i(a,\theta) \sin\theta\, P_m(\cos\theta)\, d\theta$$

$$= V_0 \int_0^{\pi/8} \sin\theta\, P_m(\cos\theta)\, d\theta$$

$\underline{P_0 = 1}$

$$I_0 = V_0 \int_0^{\pi/8} \sin\theta \, d\theta = -V_0 \cos\theta \Big|_0^{\pi/8}$$

$$= V_0 \left(1 - \cos \pi/8\right) = \frac{2 A_0}{1}$$

$$A_0 = \frac{V_0}{2}\left(1 - \cos \pi/8\right) \qquad +5$$

$P_1 = \cos\theta$

$$I_1 = V_0 \int_0^{\pi/8} \cos\theta \sin\theta \, d\theta$$

$$u = \cos\theta$$

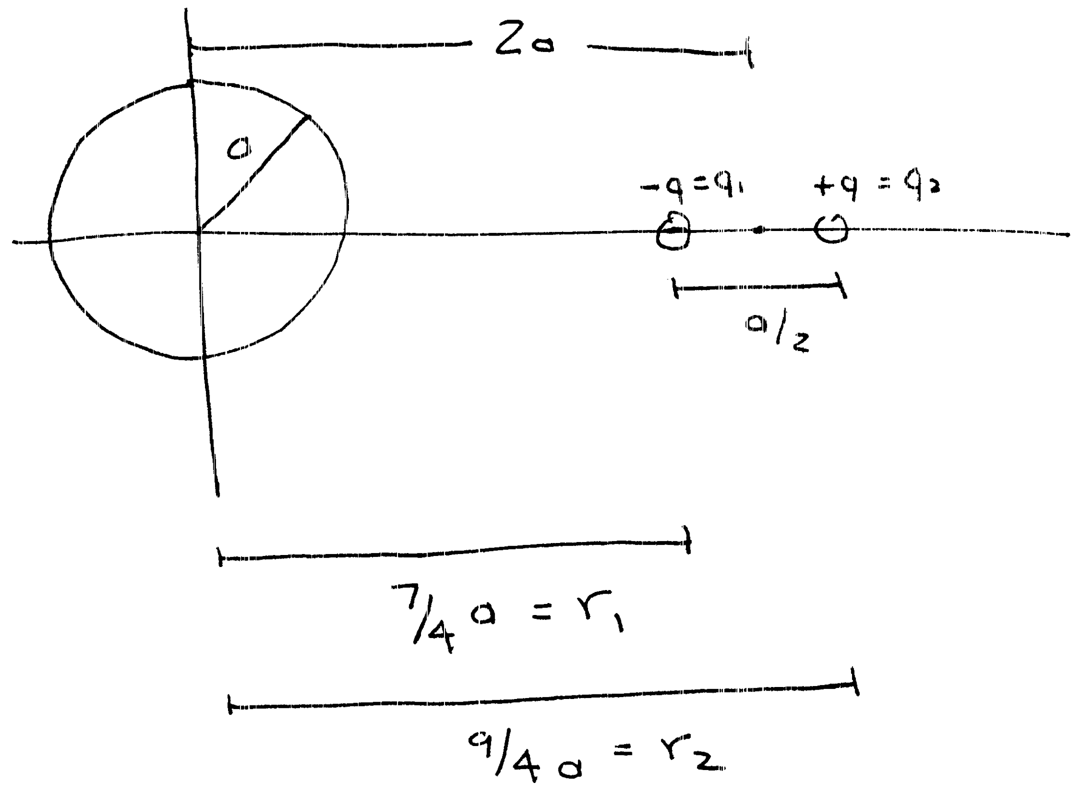
$$du = -\sin\theta \, d\theta$$

$$I_1 = -V_0 \int_1^{\cos \pi/8} u \, du = -\frac{V_0}{2}\left(\cos^2\frac{\pi}{8} - 1\right)$$

$$= \frac{V_0}{2}\left(1 - \cos^2 \pi/8\right) = \frac{2 A_1 a}{3}$$

$$A_1 = \frac{3}{4}\frac{V_0}{a}\left(1 - \cos^2 \pi/8\right) \qquad +5$$

1.2



We will require two image charges

$$q_1' = -\frac{a}{r_1} q_1 = -\frac{a}{\tfrac{7}{4}a}(-q) = \frac{4}{7} q$$

at $$r_1' = \frac{a^2}{r_1} = \frac{4}{7} a$$

+7

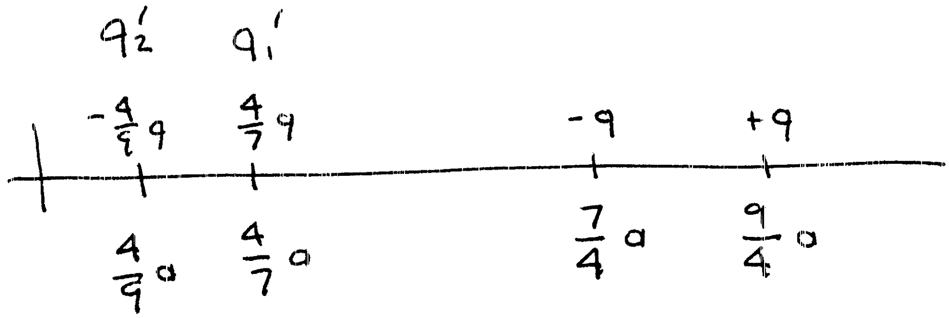
and

$$q_2' = -\frac{a}{r_2} q_2 = \frac{-a}{9/4\, a} q$$

$$= -\frac{4}{9} q$$

+7

at $$r_2' = \frac{a^2}{r_2} = \frac{4}{9} a$$



<u>Force on +q</u>

$$\vec{F}_+ = \frac{k q_1' q_2}{\left(9/4\, a - \frac{4}{7} a\right)^2} + \frac{k q_2' q_2}{\left(9/4\, a - \frac{4}{9} a\right)^2}$$

+11

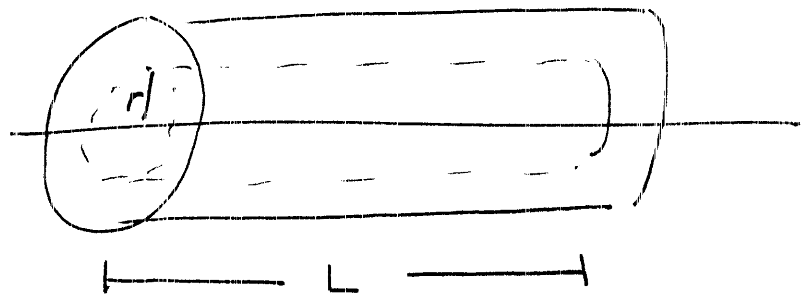
$$= k q \left(\frac{\frac{4}{7} q}{\left(\frac{47}{28} a\right)^2} - \frac{\frac{4}{9} q}{\left(\frac{65}{36} a\right)^2} \right)$$

$$\vec{F}_{+} = \frac{kq^2}{a^2}\left(\frac{4}{7}\left(\frac{28}{47}\right)^2 - \frac{4}{9}\left(\frac{36}{65}\right)^2\right)$$

$$= \frac{kq^2}{a^2}\left(0.2028 - 0.1363\right)$$

$$= 0.066\,\frac{kq^2}{d^2}$$

1.3



Use a cylindrical Gaussian surface of radius r and length L.

The charge inside the surface is $Q_{enc} = \rho V$

$$= \pi r^2 L \rho$$

The flux out of the cylinder is

$$\phi = E A_s = 2\pi r L E = \frac{Q_{enc}}{\varepsilon_0} \quad +5$$

$$E = \frac{Q_{enc}}{2\pi r L \varepsilon_0} = \frac{\pi r^2 L \rho}{2\pi r L \varepsilon_0}^{+5} = \frac{r\rho}{2\varepsilon_0} \quad +5$$

The potential difference between $r=0$ and $r=a$ is

$$\Delta V_{0a} = -\int_0^a E\,dr^{\,+5} = -\frac{\rho r^2}{4\varepsilon_0}\Big|_0^a = -\frac{\rho a^2}{4\varepsilon_0} \quad +5$$

where the sign is correct because the potential decreases in the direction of the field.

1.4 $\sigma = \sigma_0 \left(\sin^2\phi - \frac{1}{2} \right)$

Trig identity $\sin^2\phi = \frac{1}{2} - \frac{1}{2}\cos 2\phi$

$$\sigma = -\frac{\sigma_0}{2}\cos 2\phi$$

Discard terms that blow up at ∞. The potential outside the cylinder is

$$V^o = \sum A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi \qquad + \cancel{10}\ 5$$

$$\left.\frac{\partial V^o}{\partial s}\right|_a = \sum -n A_n a^{-(n+1)} \cos n\phi - n B_n a^{-(n+1)} \sin n\phi \qquad +5$$

Electrostatic Boundary Condition

$$\left.\frac{\partial V^o}{\partial s}\right|_a - \underbrace{\left.\frac{\partial V^i}{\partial s}\right|_a}_{=0} = -\frac{\sigma}{\varepsilon_0} \qquad +5$$

$$\sigma = -\frac{\sigma_0}{2}\cos 2\phi = \varepsilon_0 \sum n\, a^{-(n+1)}\left[A_n \cos n\phi + B_n \sin n\phi\right]$$

By orthogonality, only $n = 2$ is non-zero for A and all B are zero.

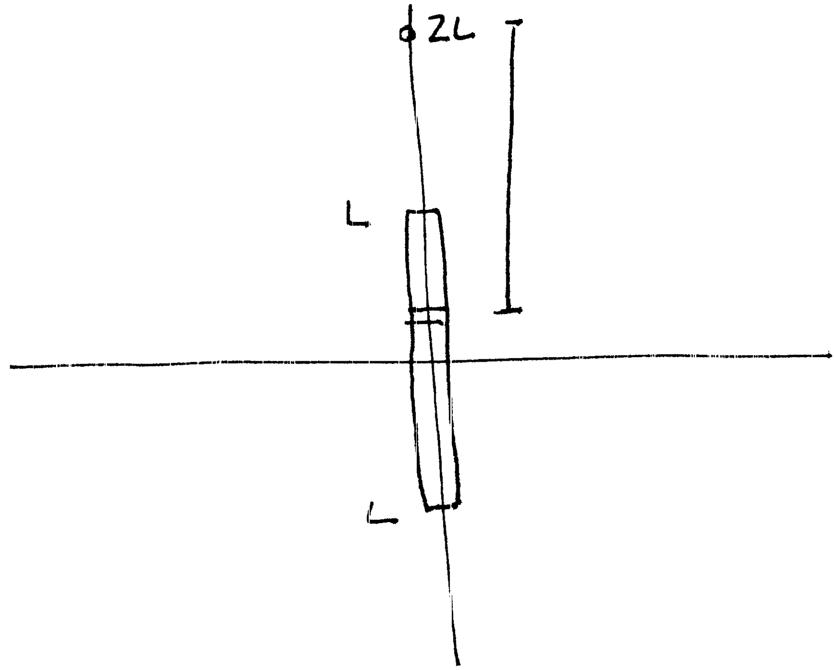
$$-\frac{\sigma_0}{2} = \varepsilon_0\, 2\, a^{-(2+1)} A_2$$

$$A_2 = -\frac{\sigma_0 a^3}{4\varepsilon_0} \quad +5$$

The potential outside the cylinder is

$$V(s,\phi) = -\frac{\sigma_0 a^3}{4\varepsilon_0 s^2}\cos 2\phi \quad +5$$

1.5



$d = 2L - z$

Coulomb's Law

$$\vec{E} = \int_{-L}^{L} \frac{k\,dq}{r''^2}\,\hat{r}'' \qquad +5$$

$\hat{r}'' = \hat{z}$

$r'' = d$

$dq = \lambda\,dz$

$$\vec{E} = \hat{z}\int_{-L}^{L} \frac{k\lambda\,dz}{(2L-z)^2} \qquad +10$$

$u = 2L - z \qquad du = -dz$

$$\vec{E} = -\hat{z}\int_{3L}^{L} \frac{k\lambda\,du}{u^2} = k\lambda\hat{z}\left(\frac{1}{u}\right)_{3L}^{L}$$

$$\vec{E} = k\lambda\hat{z}\left(\frac{1}{L} - \frac{1}{3L}\right)$$

$$= \frac{2}{3}\frac{k\lambda}{L}\hat{z} \qquad +10$$

1.6 $V = V_0(x^2 - y^2)$

$$= V_0(s^2\cos^2\phi - s^2\sin^2\phi)$$

$$= V_0 s^2(\cos^2\phi - \sin^2\phi)$$

$$= V_0 s^2 \cos 2\phi \quad \text{(Trig identity wiki)}$$

Electric field

$$\vec{E} = -\nabla V = -V_0\left(\frac{\partial s^2\cos 2\phi}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial}{\partial\phi} s^2\cos 2\phi\,\hat{\phi}\right)$$

$$= -2V_0 s\cos 2\phi\,\hat{s} + 2V_0 s\sin 2\phi\,\hat{\phi}$$

+15

Charge Density $\nabla\cdot\vec{E} = \rho/\varepsilon_0$

$$\rho = \varepsilon_0 \nabla\cdot\vec{E}$$

$$= 2V_0\varepsilon_0\left[\frac{1}{s}\frac{\partial}{\partial s}\left(-s^2\cos 2\phi\right) + \frac{1}{s}\frac{\partial}{\partial\phi}\left(s\sin 2\phi\right)\right]$$

$$= 2V_0\varepsilon_0\left[-2\cos 2\phi + 2\cos 2\phi\right] = 0$$

+10

Try Cartesian

$$-\nabla V = \vec{E} = -2V_0(x\hat{x} - y\hat{y})$$

Charge

$$\varepsilon_0 \nabla \cdot \vec{E} = -2V_0 \varepsilon_0 (1-1) = 0$$