Review Lecture

Ex Estimate the classical radius of an electron by setting its relativistic energy

$U = mc^2$, $m_e = 9.11\times10^{-31}\,\text{kg}$, $c = 3.00\times10^8\,\text{m/s}$

equal to electrostatic energy stored in its field under the assumption that the electron's charge $q_e = -1.602\times10^{-19}\,\text{C}$ is on the surface of a sphere of radius a.

Sln Find energy by integrating energy density

$$U = \int_{space} \frac{1}{2}\varepsilon_0 E^2 \, d\tau$$

$$E = \frac{k q_e}{r^2}$$

$$U = \left(\frac{1}{2}\varepsilon_0\right)\left(\frac{1}{4\pi\varepsilon_0}\right)^2 q_e^2 \int_a^\infty dr \int_0^{2\pi} d\phi \int_0^\pi d\theta \; r^2 \sin\theta \left(\frac{1}{r^2}\right)^2$$

$$= \frac{q_e^2}{32\varepsilon_0\pi^2} \cdot 4\pi \int_a^\infty \frac{dr}{r^2}$$

$$= \frac{q_e^2}{8\varepsilon_0\pi}\left(-\frac{1}{r}\right)_a^\infty = \frac{q_e^2}{8\varepsilon_0\pi a}$$

Check using capacitance

$$C = 4\pi\varepsilon_0 a$$

$$U = \frac{1}{2}\frac{q_e^2}{C} = \frac{q_e^2}{8\pi\varepsilon_0 a}$$

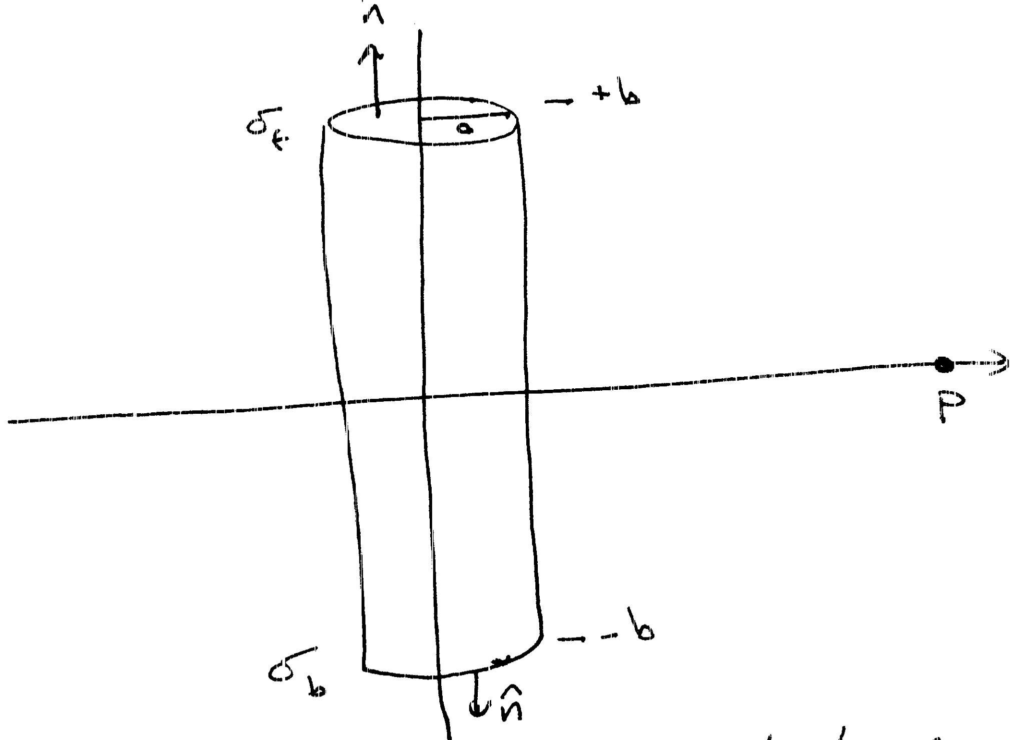
$$mc^2 = \frac{q_e^2}{8\pi\varepsilon_0 a}$$

$$a = \frac{q_e^2}{8\pi\varepsilon_0 mc^2} = 1.4 \times 10^{-15}\ m$$

(about the size of a proton).

Ex A cylinder of radius a and length $2b$ is centered on the origin. The axis of the cylinder is the z-axis. The cylinder contains a material with polarization $\vec{P} = -\frac{1}{2}\gamma z^2 \hat{z}$

Compute the field at a point $10b$ along the x-axis (approximately). $b >> a$.



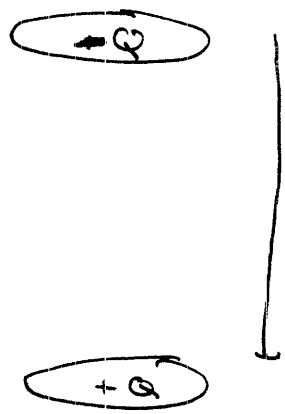
Find approximate field using dipole moment.

$$\rho_b = -\nabla \cdot \vec{P} = \gamma z$$

$$\sigma_b = \vec{P} \cdot (-\hat{z}) = \frac{1}{2}\gamma b^2$$

$$\sigma_t = \vec{P} \cdot (\hat{z}) = -\frac{1}{2}\gamma b^2$$

The top and bottom form a simple dipole



$$Q = \sigma_b \pi a^2 = \frac{1}{2}\gamma b^2 \cdot \pi a^2 = \frac{1}{2}\pi \gamma a^2 b^2$$

$$\vec{p} = -Q d \hat{z} = -\left(\frac{1}{2}\pi\gamma a^2 b^2\right)(2b)\,\hat{z}$$

$$= -\pi \gamma a^2 b^3 \,\hat{z}$$

The dipole moment of the volume bound charge is

$$\vec{p} = \int_{\text{cylinder}} \vec{r}\,'\rho(\vec{r}\,')\,d\tau'$$

The dipole moment evidently points in the $\hat{z}$ directions, so work on the z component of ρ.

$$P_z = \int z' \rho_b(z')\, d\tau'$$

$$= \int z' (+\gamma z')\, d\tau'$$

$$= \gamma \int z'^2\, d\tau' \qquad d\tau' = dz'\, ds'\, s'\, d\phi'$$

$$= \gamma \int_{-b}^{b} dz' \int_0^{2\pi} d\phi' \int_0^a ds'\; s' z'^2$$

$$= 2\pi\gamma \cdot \frac{a^2}{2} \cdot \int_{-b}^{b} dz'\, z'^2$$

$$= \pi\gamma a^2 \left(\frac{z'^3}{3}\right)_{-b}^{b} = \frac{2}{3}\pi\gamma a^2 b^3$$

$$\vec{p}_{bulk} = \frac{2}{3}\pi\gamma a^2 b^3\, \hat{z}$$

Total dipole moment

$$\vec{p} = \vec{p}_{bulk} + \vec{p}_{surface} = \frac{2}{3}\pi\gamma a^2 b^3\, \hat{z} - \pi\gamma a^2 b^3\, \hat{z}$$

$$= -\frac{1}{3}\pi\gamma a^2 b^3\, \hat{z}$$

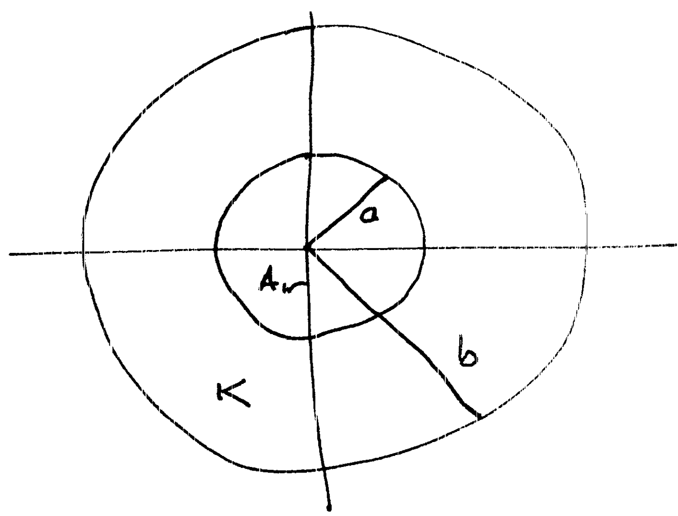
Field at a point on x-axis

$$\vec{E} = -\frac{pk}{x^3}\,\hat{z}$$

$$= -\frac{pk}{(10b)^3}\,\hat{z} = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{1000b^3}\cdot\left(-\frac{1}{3}\pi\gamma a^2 b^3\right)$$

$$= \frac{1}{12{,}000}\cdot\frac{\gamma a^2}{\varepsilon_0}\,\hat{z}$$

Ex The surface of a thick dielectric cylinder is held at $V(b, \phi) = V_0 \sin 2\phi$. The inner radius of the shell is a and the shell has dielectric constant $\kappa = \varepsilon_r$. Find the bound charge distribution on the inner surface of the shell.



Solve Laplace's Eqn

For $s < a$

$$V_{air} = \sum_n A_n \sin n\phi \, s^n$$

For $a < s < b$

$$V_\kappa = \sum_n B_n \sin n\phi \, s^n + C_n \sin n\phi \, s^{-n}$$

where I've thrown away $\cos n\phi$ solutions because BC depends only on $\sin n\phi$.

Continuity BC

$$V_\kappa(b,\phi) = V_0 \sin 2\phi$$

$$V_{air}(a,\phi) = V_\kappa(a,\phi)$$

ES BC

$$\varepsilon_\kappa \left.\frac{\partial V_\kappa}{\partial s}\right|_a - \varepsilon_{air} \left.\frac{\partial V_{air}}{\partial s}\right|_a = -\sigma_f = 0$$

$$\varepsilon_\kappa = \kappa \varepsilon_0 \qquad \varepsilon_{air} = \varepsilon_0$$

$$\kappa \left.\frac{\partial V_\kappa}{\partial s}\right|_a - \left.\frac{\partial V_{air}}{\partial s}\right|_a = 0$$

Apply BC

$$V_\kappa(b,\phi) = V_0 \sin 2\phi = \sum B_n \sin n\phi \, b^n + C_n \sin n\phi \, b^{-n}$$

$$\Rightarrow B, C = 0 \quad n \neq 2$$

$$V_0 = B_2 b^2 + C_2 b^{-2}$$

$$V_{air}(a,\phi) = V_K(a,\phi)$$

$$\sum A_n a^n \sin n\phi = B_2 a^2 \sin 2\phi + \frac{C_2}{a^2} \sin 2\phi$$

Again orthogonality implies, $A_n = 0 \quad n \neq 2$

$$A_2 a^2 = B_2 a^2 + \frac{C_2}{a^2}$$

$$A_2 = B_2 + \frac{C_2}{a^4}$$

<u>Electrostatic BC</u> $\quad V_{air} = A_2 s^2 \sin 2\phi$

$$V_K = B_2 s^2 \sin 2\phi + C_2 s^{-2} \sin 2\phi$$

$$\left.\frac{\partial V_K}{\partial s}\right|_a = 2B_2 a \sin 2\phi - \frac{2C_2}{a^3} \sin 2\phi$$

$$\left.\frac{\partial V_{air}}{\partial s}\right|_a = 2A_2 a \sin 2\phi$$

$$K \left.\frac{\partial V_K}{\partial s}\right|_a - \left.\frac{\partial V_{air}}{\partial s}\right|_a = K\left(2B_2 a \sin 2\phi - \frac{2C_2}{a^3} \sin 2\phi\right)$$

$$- 2A_2 a \sin 2\phi = 0$$

$$2B_2 a k - \frac{2C_2 k}{a^3} - 2A_2 a = 0$$

$$B_2 k - \frac{C_2 k}{a^4} - A_2 = 0$$

Solve

$$B_2 + \frac{C_2}{b^4} = \frac{V_0}{b^2} \qquad (1)$$

$$A_2 - B_2 - \frac{C_2}{a^4} = 0 \qquad (2)$$

$$A_2 - B_2 k + \frac{C_2 k}{a^4} = 0 \qquad (3)$$

$$(1)+(2) \implies \quad A_2 = \frac{V_0}{b^2}$$

$$k(2) + (3) \implies$$

$$(k+1) A_2 - 2k B_2 = 0$$

$$B_2 = \frac{k+1}{2k} A_2 = \frac{k+1}{2k} \frac{V_0}{b^2}$$

$$C_2 = a^4 (A_2 - B_2)$$

$$= a^4 \left(\frac{V_0}{b^2} - \frac{k+1}{2k} \frac{V_0}{b^2} \right)$$

$$= \frac{V_0 a^4}{b^2} \left(1 - \frac{k+1}{2k} \right) =$$

$$= \frac{V_0 a^4}{b^2} \left(\frac{2k - k - 1}{2k} \right)$$

$$= \frac{V_0 a^4}{b^2} \frac{k-1}{2k}$$

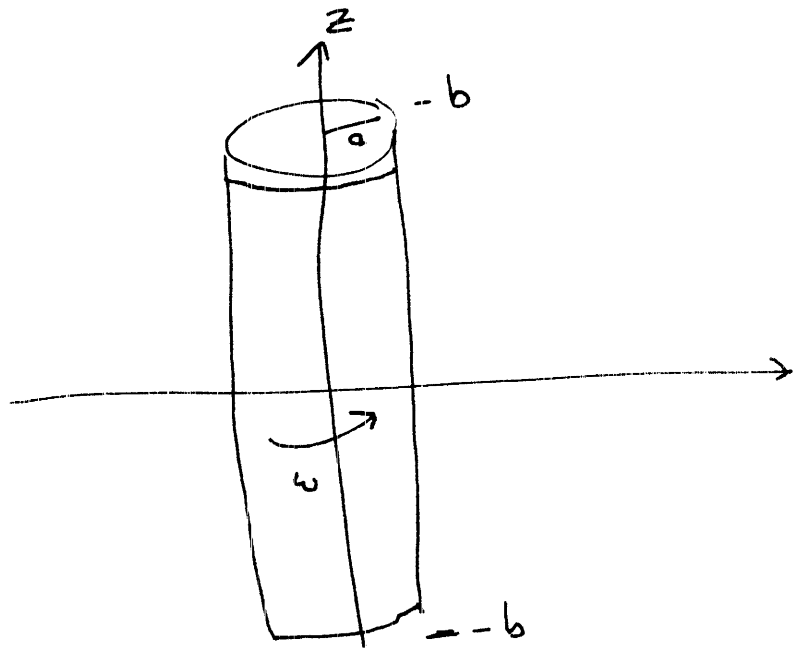
Charge Density at Surface

Use Pill box,

$$\Phi_e = \left(\vec{E}_k \cdot \hat{s} + \vec{E}_{air} \cdot (-\hat{s}) \right) A = \frac{\sigma A}{\varepsilon_0}$$

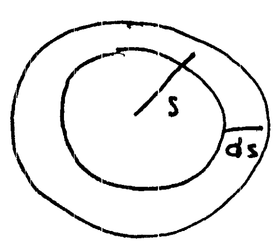
$$\left. \frac{\partial V_k}{\partial s} \right|_a - \left. \frac{\partial V_{air}}{\partial s} \right|_a = \frac{\sigma}{\varepsilon_0}$$

__Ex__ A cylinder with charge density $\rho = \gamma z$ rotates with angular velocity ω about its axis. Compute field at a point ~~far~~ far from the cylinder along z-axis.



__Compute Moment of Ring__

$$\vec{J} = \rho \vec{v} = \rho \omega s \hat{\phi}$$



$$d\vec{I} = \vec{J} \cdot da = \vec{J}\, dz\, ds$$

The moment of the ring is

$$dm = dI\,A = \pi s^2\, dI = \pi s^2\, J\, ds\, dz$$

The total moment is

$$m = \int dm = \int_{b}^{b} dz \int_0^a ds \left(\pi s^2\right)\left(\underset{\uparrow\ \gamma z}{\rho}\, \omega s\right)$$

$$\cancel{= \pi \rho}$$

$$m = \pi \omega \int_{-b}^{b} dz \int_0^a ds\ s^2 \cdot s \cdot \gamma z$$

$$= \pi \omega \gamma \int_{-b}^{b} dz\, z \int_0^a ds\, s^3 = 0$$

$$= \cancel{\pi \omega}\, \gamma$$

Change problem so cylinder only extends from 0 to b.

$$m = \pi \omega \gamma \underbrace{\int_0^b dz\, z}_{\frac{b^2}{2}} \underbrace{\int_0^a ds\, s^3}_{\frac{a^4}{4}} = \frac{\pi \omega \gamma b^2 a^4}{8}$$

Field at a Point Along Axis

$$E = 2\frac{\mu_0}{4\pi}\frac{m}{z^3}\hat{z}$$

Ex ~~If~~ Is $\vec{B} = \gamma z^2 \hat{z}$ a valid ~~electrostatic~~ magnetostatic field? If so find the current that produced the field?

Sln $\nabla \cdot \vec{B} = 2\gamma z \neq 0$ not ~~a valid~~ valid field.

What about $\vec{B} = \gamma z^2 \hat{\phi}$?

$\nabla \cdot \vec{B} = 0$ ✓

$$\nabla \times \vec{B} = -\frac{\partial B_\phi}{\partial z}\hat{s} + \frac{1}{s}\frac{\partial (B_\phi s)}{\partial s}$$

$$= -\gamma z \hat{s} + \frac{\gamma z^2}{s}\hat{z} = \mu_0 \vec{J}$$

$$\vec{J} = \frac{1}{\mu_0}\left[-\gamma z \hat{s} + \frac{\gamma z^2}{s}\hat{z}\right]$$

Check static current

$$\nabla \cdot \vec{J} = \frac{1}{\mu_0}\left[\frac{1}{s}\frac{\partial}{\partial s} s J_s + \frac{\partial J_z}{\partial z}\right]$$

$$\nabla \cdot \vec{J} = \frac{1}{\mu_0}\left[\frac{-\gamma z}{s} + \frac{2\gamma z}{s} \right] \neq 0$$

not static