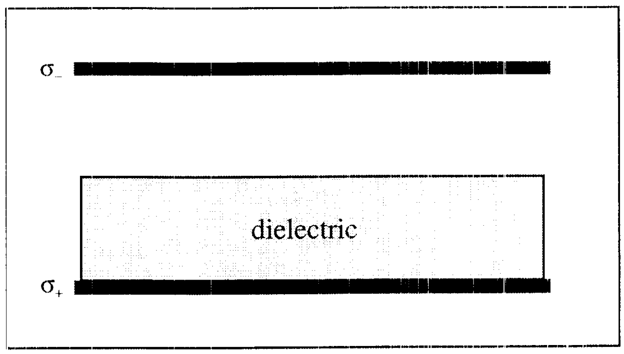
Electricity and Magnetism - Test 2 - Spring 2010

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

Problem 2.1 A system of current has a vector potential of $\vec{A} = \gamma s^2 \hat{\phi}$ in cylindrical coordinates. Find the current that could have resulted in $\vec{A}$. Is the current a valid magnetostatic current?

Problem 2.2 The infinite rectangular region $-a < z < a$ contains volume current density $\vec{J} = \gamma z \hat{x}$ where gamma is a constant. Find the magnetic field everywhere.

Problem 2.3 A linear dielectric slab with dielectric constant ϵ_r is placed between two infinite parallel planes of charge with charge density $\pm\sigma$. Find $\vec{D}$, $\vec{E}$, $\vec{P}$, and ρ_b in the dielectric, and the bound charge density on the top and bottom surface of the dielectric.



Problem 2.4 A potential of $V_0 \cos(\theta)$ is established on the inner surface of a spherical dielectric with inner radius a and outer radius b. The dielectric constant of the material is ϵ_r. Find the potential for $r > a$. You may report a system of equations that needs to be solved to find the coefficients of the potential functions. Actually solving these equations turns out to be quite messy. These equations should be a set of simple linear, non-differential equations.

Problem 2.5 A disk with surface charge density γ/s, inner radius a and outer radius b is spun at angular velocity ω about an axis through its center. Find the magnetic field at the center of the disk.

Problem 2.6 A spherical system has polarization $\vec{P} = \gamma r^2 \hat{r}$ for radius $r < a$ and $\vec{P} = 0$ for $r > a$. Find the electric field everywhere.

2.1

$\vec{A} = \gamma s^2 \hat{\phi}$ cylindrical

$$\vec{B} = \nabla \times \vec{A}$$

$$= -\frac{\partial A_\phi}{\partial z}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}\left(s A_\phi\right)\hat{z}$$

$$\vec{B} = \gamma \frac{1}{s}\frac{\partial}{\partial s} s^3 \hat{z}$$

$$\vec{B} = 3\gamma s \hat{z}$$

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B}$$

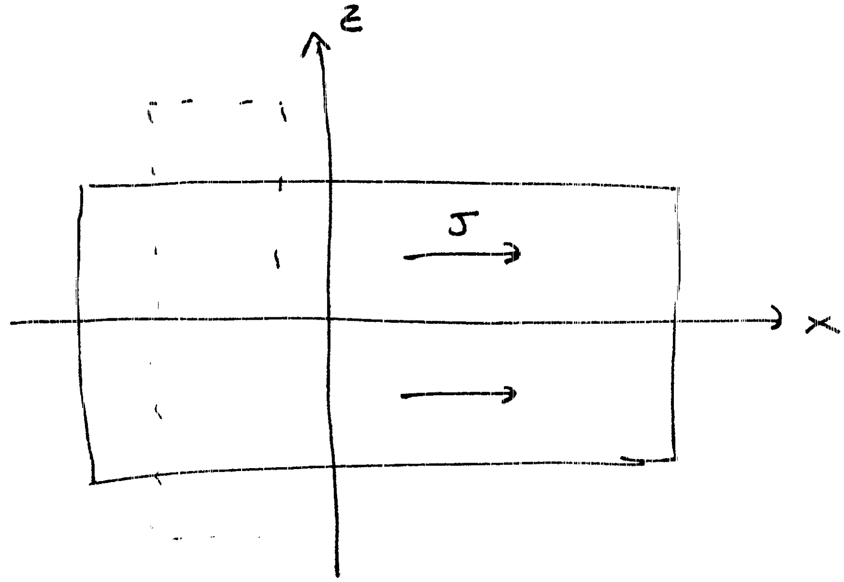
$$= \frac{1}{\mu_0}\left[\frac{1}{s}\frac{\partial B_z}{\partial \phi}\hat{s} - \frac{\partial B_z}{\partial s}\hat{\phi}\right]$$

$$= -\frac{1}{\mu_0} \cdot 3\gamma \hat{\phi} = -\frac{3\gamma}{\mu_0}\hat{\phi}$$

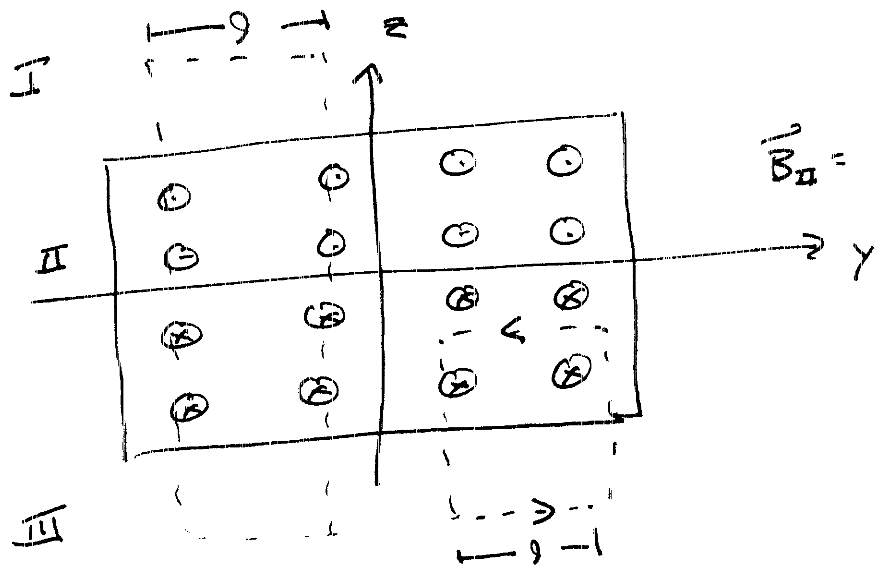
To be valid magnetostatic current, $\nabla \cdot \vec{J} = 0$

$$\nabla \cdot \vec{J} = \frac{1}{s}\frac{\partial J_\phi}{\partial \phi} = 0$$ ✓ yes.

2.2



<u>End View</u>



$\vec{B}_{II} = B_{II}\,\hat{y}$

Path 1 Path 2

<u>Path 1</u> Current encircled $= 0$. Since field of sheet of current does not fall off with distance

$$\vec{B}_{I} = \vec{B}_{III} = 0$$

Path 2

Current encircled

$$I_{enc} = \ell \int_{-a}^{z} J \, dz$$

$$= \ell \gamma \int_{-a}^{z} z$$

$$= \frac{\gamma \ell}{2} \left(z^2 - a^2 \right)$$

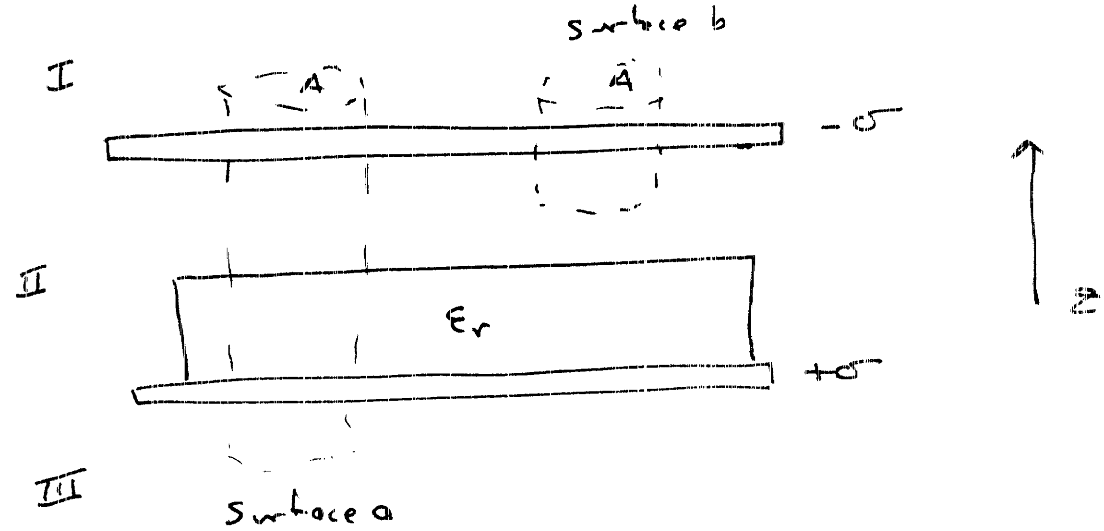
Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$-B_{II} \ell + B_{I} \ell = \mu_0 I_{enc}$$

$$\vec{B}_{II} = -\frac{\mu_0 \gamma}{2} \left(z^2 - a^2 \right) \hat{y}$$

2.3



Surface a $\quad Q_{fenc} = 0$

$$\Phi_d = D_I A - D_{III} A = 0 \qquad \text{Gauss (Displacement)}$$

By symmetry, fields equal but opposite

$$D_I = -D_{III} \implies \vec{D}_I = \vec{D}_{III} = 0$$

Surface b $\quad Q_{fenc} = -\sigma A$

$$D_I A - D_{II} A = Q_{fenc} = -\sigma A$$

$$\vec{D}_{II} = \sigma \hat{z} = \vec{D} \text{ inside dielectric.}$$

Since dielectric linear,

$$\varepsilon_0 \varepsilon_r \vec{E}_{\pi} = \vec{D} = \sigma \hat{z}$$

$$\vec{E}_{\pi} = \frac{\sigma}{\varepsilon_0 \varepsilon_r} \hat{z}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$$

$$= \varepsilon_0(\varepsilon_r - 1)\frac{\sigma}{\varepsilon_0 \varepsilon_r}\hat{z} = \frac{\varepsilon_r - 1}{\varepsilon_r}\sigma\hat{z}$$

Bound Charge Density

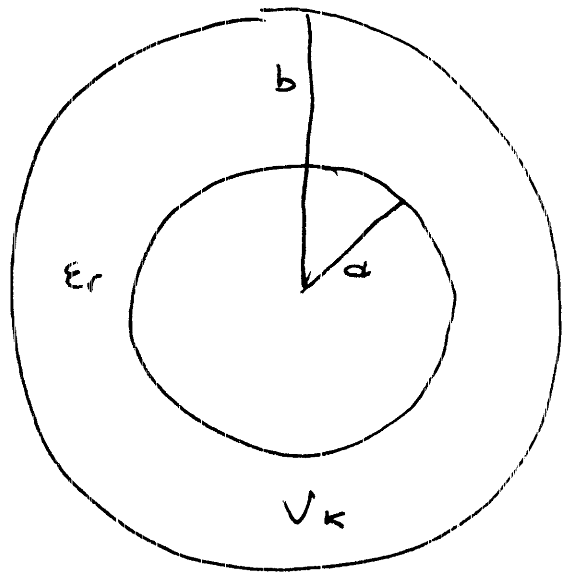
$$\rho_b = -\nabla \cdot \vec{P} = 0$$

<u>Surface Charge Densities</u>

Top surface $\sigma_t = \hat{z} \cdot \vec{P} = \dfrac{\varepsilon_r - 1}{\varepsilon_r}\sigma$

Bottom surface $\sigma_b = (-\hat{z}) \cdot \vec{P} = \dfrac{-(\varepsilon_r - 1)}{\varepsilon_r}\sigma$

2.4



Solution to Laplaces Eqn keep terms that don't explode.

$$V_K = \sum_n \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\cos\theta)$$

$$V_o = \sum_n C_n r^{-(n+1)} P_n(\cos\theta)$$

The applied potential is $V_o P_1(\cos\theta)$ so only $n=1$ terms are needed.

$$V_K = \left[A_1 r + \frac{B_1}{r^2}\right] P_1(\cos\theta)$$

$$V_o = \frac{C_1}{r^2} P_1(\cos\theta)$$

Boundary Conditions

$$V_k(a, \phi) = V_0 P_1(\cos\theta)$$

$$= \cancel{E_0}\left(A_1 a + \frac{B_1}{a^2}\right) P_1(\cos\theta)$$

$$V_0 = A_1 a + \frac{B_1}{a^2}$$

Continuity at b

$$V_k(b, \phi) = V_0(b, \phi)$$

$$\left(A_1 b + \frac{B_1}{b^2}\right) P_1(\cos\theta) = \frac{C_1}{b^2} P_1(\cos\theta)$$

$$A_1 b + \frac{B_1}{b^2} = \frac{C_1}{b^2}$$

Electrostatic BC

$$\varepsilon_0 \frac{\partial V_0}{\partial r}\Big|_b - \varepsilon_r \varepsilon_0 \frac{\partial V_k}{\partial r}\Big|_b = -\sigma_f = 0$$

$$-2\frac{C_1}{b^3} - \varepsilon_r\left(A_1 - 2\frac{B_1}{b^3}\right) = 0$$

$$V_0 a^2 = A_1 a^3 + B_1 \qquad (1)$$

$$0 = A_1 b^3 + B_1 - C_1 \qquad (2)$$

$$0 = A_1 b^3 \varepsilon_r - 2\varepsilon_r B_1 + 2C_1 \qquad (3)$$

2 (2) + (1)

$$A_1 b^3 (2+\varepsilon_r) + (2-2\varepsilon_r) B_1 = 0$$

$$B_1 = -\frac{A_1 b^3}{2}\frac{2+\varepsilon_r}{1-\varepsilon_r} = \frac{A_1 b^3}{2}\frac{2+\varepsilon_r}{\varepsilon_r - 1} \qquad (4)$$

(4) → (1)

$$V_0 a^2 = A_1 a^3 + \frac{A_1 b^3}{2}\frac{2+\varepsilon_r}{\varepsilon_r - 1}$$

$$= A_1\left(a^3 + \frac{b^3}{2}\frac{2+\varepsilon_r}{\varepsilon_r - 1}\right)$$

$$A_1 = \frac{V_0 a^2}{a^3 + \frac{b^3}{2}\frac{2+\varepsilon_r}{\varepsilon_r - 1}} \qquad (5)$$

(5) $\rightarrow$ (4)

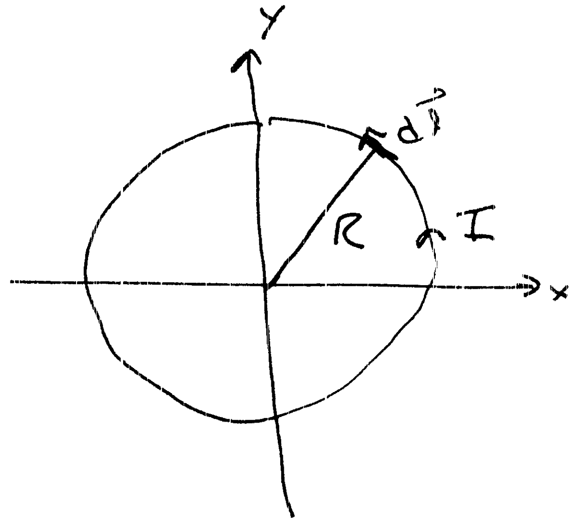
$$B_1 = \frac{A_1 b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1}$$

$$= \frac{V_0 a^2}{a^3 + \frac{b^3}{2}\left(\frac{2+\varepsilon_r}{\varepsilon_r - 1}\right)} \cdot \frac{b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1}$$

$$C_1 = A_1 b^3 + B_1$$

$$= \frac{V_0 a^2 b^3}{a^3 + \frac{b^3}{2}\left(\frac{2+\varepsilon_r}{\varepsilon_r - 1}\right)} + \frac{V_0 a^2 b^3}{a^3 + \frac{b^3}{2}\left(\frac{2+\varepsilon_r}{\varepsilon_r - 1}\right)} \cdot \frac{2+\varepsilon_r}{2(\varepsilon_r - 1)}$$

(2.5) Magnetic field of a ring of current at its center.



$$\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l} \times \hat{r}''}{r''^2}$$

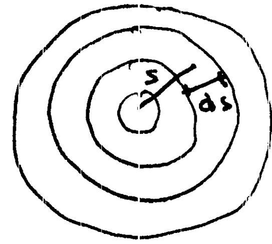
$$d\vec{l} \times \hat{r}'' = dl$$

since $d\vec{l} \perp \hat{r}''$ and $|\hat{r}''| = 1$.

$$\vec{B} = \frac{\mu_0 I}{4\pi R^2}\int_c dl = \frac{\mu_0 I\, 2\pi R}{4\pi R^2}$$

$$= \frac{\mu_0 I}{2R}$$

Sum contributions of different loops.



Current Density $\quad \vec{K} = \sigma \omega s \hat{\phi} = \gamma \omega \hat{\phi}$

Current Flowing in ring of thickness ds

$$dI = K\, ds = \gamma \omega\, ds$$

Field of Ring

$$dB = \frac{\mu_0\, dI}{2s} = \frac{\mu_0 \gamma \omega}{2s}\, ds$$

Total Field

$$\vec{B} = \int_a^b \frac{\mu_0 \gamma \omega}{2s}\, ds$$

$$= \frac{\mu_0 \gamma \omega}{2} \ln\left(b/a\right)$$

(2.6) $\vec{P} = \gamma r^2 \hat{r}$

$$\rho_b = -\nabla \cdot \vec{P} = -\gamma \frac{1}{r^2} \frac{\partial}{\partial r} r^4$$

$$= -4\gamma r$$

Surface charge $\sigma_b = \vec{P} \cdot \hat{r} = \gamma a^2$

Inside Object $r < a$ Gaussian surface radius r

$$Q_{enc} = \int \rho \, d\tau$$

$$= \int \rho \, r^2 \sin\theta \, dr \, d\phi \, d\theta$$

$$= 4\pi \int_0^r \rho \, r^2 \, dr$$

$$= -16\gamma\pi \int_0^r r^3 \, dr$$

$$= -4\gamma\pi r^4$$

Gauss Law

$$\Phi_e = \int \vec{E} \cdot \hat{n} \, dA = 4\pi r^2 E = \frac{Q_{enc}}{\epsilon_0}$$

$$\vec{E}_i = \frac{Q_{enc}}{4\pi\epsilon_0 r^2} \hat{r} = -\frac{4\gamma\pi r^4}{4\pi\epsilon_0 r^2} \hat{r}$$

$$= -\frac{\gamma r^2}{\epsilon_0} \hat{r}$$

Field Outside   Charge enclosed is total charge of volume charge plus surface charge.

$$Q_{enc} = Q_{vol} + Q_{surface}$$

$$= -4\gamma\pi a^4 \quad + (\gamma a^2)(4\pi a^2)$$

$$= 0$$

$$\vec{E}_o = 0$$

An alternate, but cool, solution I didn't think of

$$Q_{f\,enc} = 0 \quad \text{everywhere}$$

by symmetry $\Longrightarrow$ $\vec{D} = 0$ everywhere.

<u>Outside</u> $\quad \vec{D} = \varepsilon_0 \vec{E}_o \quad \Longrightarrow \quad \vec{E}_o = 0$

<u>Inside</u> $\quad \vec{D} = \varepsilon_0 \vec{E}_i + \vec{P} = 0$

$$\vec{E}_i = -\frac{\vec{P}}{\varepsilon_0} = \frac{-\gamma r^2}{\varepsilon_0}\,\hat{r}$$