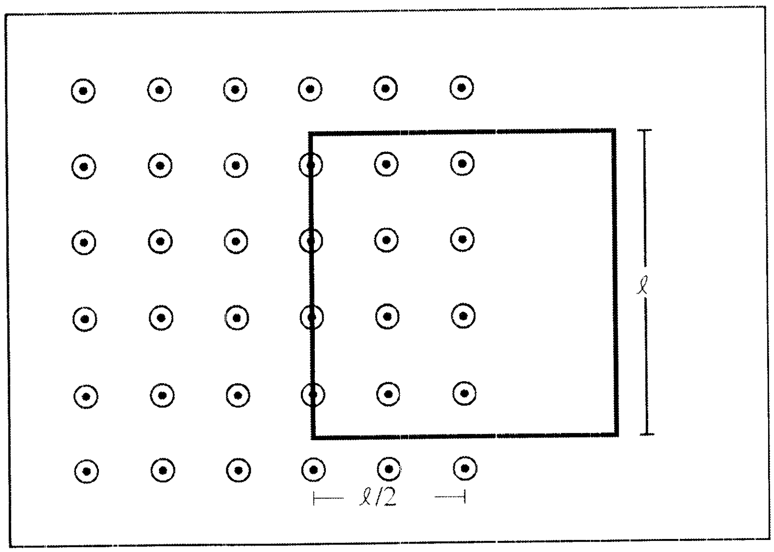
Electricity and Magnetism - Test 3 - Spring 2010

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

Problem 3.1 Consider two concentric strongly conducting spherical shells where the smaller has outer radius a and the larger inner radius c. The volume between the shells is filled with two weakly conducting materials. The space from $r = a$ to $r = b$ contains a material with conductivity σ_1. The space from $r = b$ to $r = c$ contains a material with conductivity σ_2. Compute the resistance between the shells (between $r = a$ and $r = c$). Note. $a < b < c$.

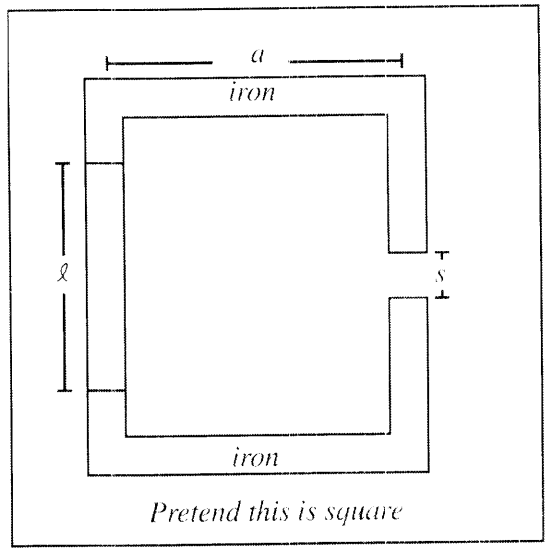
Problem 3.2 An aluminum square with resistivity $2.65 \cdot 10^{-8}\Omega$m is placed halfway inside a magnetic field. The magnetic field is being turned off and the magnitude of the field obeys $B_0 e^{-t/\tau}$ where $\tau = 2$s and $B_0 = 0.2$T. The aluminum square has side length $\ell = 2$cm and cross-sectional area $A = 1cm^2$. Compute the force exerted on the loop at time $t = 0$. Does the force tend to push the loop out of the field or draw the loop into the field?



Problem 3.3 As part of an honors project in UPII this semester, a student used a stack of NdFeB magnets to power a hand generator. Each magnet was a cylinder of height $h = 1$mm and radius $r = 1$cm. The student made the approximation that the magnetic field of a stack of six magnets was six times the magnetic field of one. The magnetization of NdFeB is $1.02 \cdot 10^6$A/m. Compare the field at the center of a single magnet with the field at the center of a stack of six magnets. You may model the bound current of a single magnet as a ring, but may not use this model for the stack. You may use the formula for the field of the current distribution of the stack if you included it on your formula card, if not you will have to re-derive it.

Problem 3.4 The region between $z = -a$ and $z = +a$ contains a changing electric field $\vec{E} = E_0 \sin(\omega t)\hat{x}$. Compute the magnetic field at points $z > a$.

Problem 3.5 A horseshoe magnet is formed into a square. It is made using a section of NdFeB permanent magnetic material with magnetization density $1.02 \cdot 10^6$ A/m of length $\ell = 2$cm. The square has average side length $a = 3$cm and a gap with width $s = 1$cm. Compute the magnetic field in the gap. The relative permeability of iron is 100. Note, this problem may be worked in the same manner as the other magnets formed of rings of various materials worked in the homework.

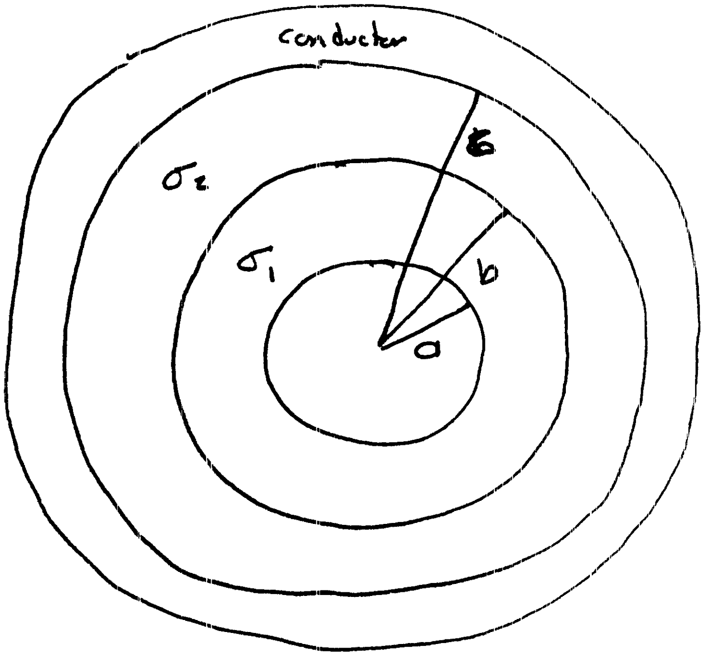


Problem 3.6 A cylindrical stainless steel channel with relative permeability μ_r, inner radius a and outer radius b, confines a plasma with current density

$$\vec{J} = \frac{J_0 a}{s} \sin\left(\frac{\pi s}{a}\right) \hat{z}$$

where the axis of the solenoid is the z-axis and J_0 is a constant. Compute $\vec{H}$, $\vec{B}$, and $\vec{M}$ everywhere.

3.1



Assume I flows through all cross sections. The current density is

$$\vec{J} = \frac{I}{4\pi r^2}\hat{r}$$

By Ohm's Law, the conductivity is related to the current by

$$\vec{J} = \sigma_1 \vec{E}$$

$$\vec{E}_1 = \frac{\vec{J}}{\sigma_1} = \frac{I}{4\pi r^2 \sigma_1}\hat{r}$$

The potential difference between the shell and radius b is then

$$\Delta V_1 = -\int \vec{E}_1 \cdot d\vec{l} = -\frac{I}{4\pi\sigma_1}\int_a^b \frac{dr}{r^2}$$

$$\Delta V_1 = \frac{I}{4\pi\sigma_1}\left(\frac{1}{b} - \frac{1}{a}\right)$$

which is correctly negative.

~~The resistance~~ Likewise, the potential difference across the second conductor is

$$\Delta V_2 = \frac{I}{4\pi\sigma_2}\left(\frac{1}{c} - \frac{1}{b}\right)$$

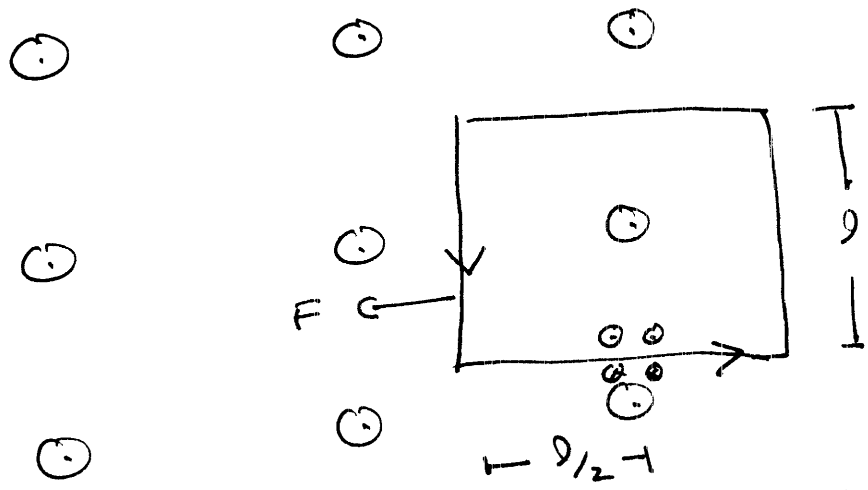
The resistance is then

$$R = \frac{|\Delta V_1 + \Delta V_2|}{I} = \frac{1}{4\pi}\left(\frac{1}{\sigma_1}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{\sigma_2}\left(\frac{1}{b} - \frac{1}{c}\right)\right)$$

Dimensions are correct

$$[R] = \left[\frac{l}{\sigma A}\right] = \left[\frac{1}{\sigma}\right]\left[\frac{1}{m}\right]$$

3.2



Field decreasing, to oppose decrease the induced current must produce field to oppose change (Lenz)
So the induced current is CCW.

The force on the top and bottom cancel, so the only force is that on the left side, $\vec{F} = I\vec{L} \times \vec{B}$. By RHR, the force is to the left of the page and the loop is drawn into the field.

The resistance of the loop is $R = \frac{\rho(4\ell)}{A}$

$$R = \frac{(2.65\times10^{-8}\,\Omega m)(4)(0.02m)}{1\times10^{-4}m^2}$$

$$= 2.12\times10^{-5}\,\Omega$$

The emf is given by Faraday's Law

$$\text{emf} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt} B\frac{\ell^2}{2}$$

$$= -\frac{\ell^2 B_0}{2}\frac{d}{dt} e^{-t/\tau} = \frac{\ell^2 B_0}{2\tau} e^{-t/\tau}$$

At $t=0$,

$$\text{emf} = \frac{\ell^2 B_0}{2\tau} = \frac{(0.02\,\text{m})^2 (0.2\,\text{T})}{2(2\,\text{s})} = 2\times 10^{-5}\,\text{V}$$

The current in the loop is then

$$I = \frac{\text{emf}}{R} = \frac{2\times 10^{-5}\,\text{V}}{2.12\times 10^{-5}\,\Omega} = 0.943\,\text{A}$$

The force is given by the Lorentz force

$$|\vec{F}| = |I\vec{\ell}\times\vec{B}| = I\ell B = (0.943\,\text{A})(0.02\,\text{m})(0.2\,\text{T})$$

$$= 0.0038\,\text{N}$$

<u>Symbolically</u>

$$F(0) = I\ell B = \left(\frac{\ell^2 B_0}{2\tau R}\right) B_0 \ell = \frac{\ell^3 B_0^{\,2}}{2\tau(\rho 4\ell/A)}$$

$$= \frac{\ell^2 B_0^{\,2} A}{8\tau\rho}$$

The emf is given by Faraday's Law

$$\text{emf} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\left(\frac{l^2}{2}\right)B$$

$$= \frac{-l^2 B_0}{2}\frac{d}{dt}e^{-t/\tau} = \frac{l^2 B_0}{2\tau}e^{-t/\tau}$$

The emf at $t=0$ is then

$$\text{emf} = \frac{l^2 B_0}{2\tau} = \frac{(0.02m)(0.2T)}{2(2s)} = 1\times10^{-3}V$$

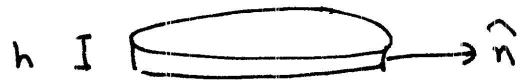
The current in the loop is then

$$I = \frac{\text{emf}}{R} = \frac{1\times10^{-3}V}{2.12\times10^{-5}\Omega} = 47.2A$$

The force is given by the Lorentz force

$$F = ILB = (47.2A)(0.02m)(0.2T) = 0.19N$$

(3.3)



The bound surface current is

$$\vec{K}_b = \vec{M} \times \hat{s} = M_0 \hat{\phi}$$

For one magnet, treat it as a ring of current with current $I = K_b h = M_0 h$

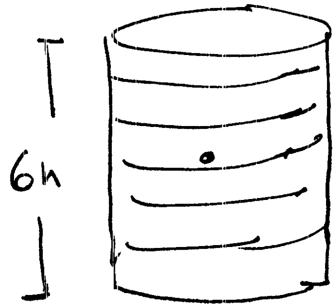
The field of a ring at the origin is

$$B_r = \frac{\mu_0 I}{2r} = \frac{\mu_0 M_0 h}{2r}$$

$$= \frac{\left(4\pi \times 10^{-7} \frac{Tm}{A}\right)\left(1.02 \times 10^{6} \frac{A}{m}\right)(0.001 m)}{2(0.01 m)}$$

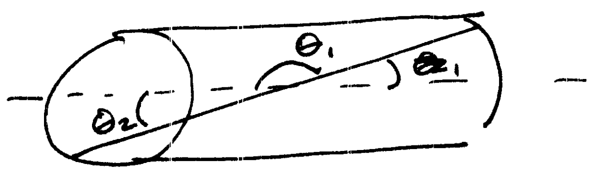
$$= 0.064 T$$

The stack of 6 magnets must be modeled as a finite solenoid



$$K_b = M_o = nI$$

The field of a finite solenoid



$$B_s = \frac{\mu_o n I}{2}\left(\cos\theta_2 - \cos\theta_1\right)$$

$$= \frac{\mu_o n I}{2}\left(\cos\theta_2 + \cos\theta_2\right)$$

$$= \mu_o n I \cos\theta_2$$

$$\cos\theta_2 = \frac{3h}{\sqrt{(3h)^2 + r^2}} = \frac{3(0.001m)}{\sqrt{(0.003m)^2 + (0.01m)^2}}$$

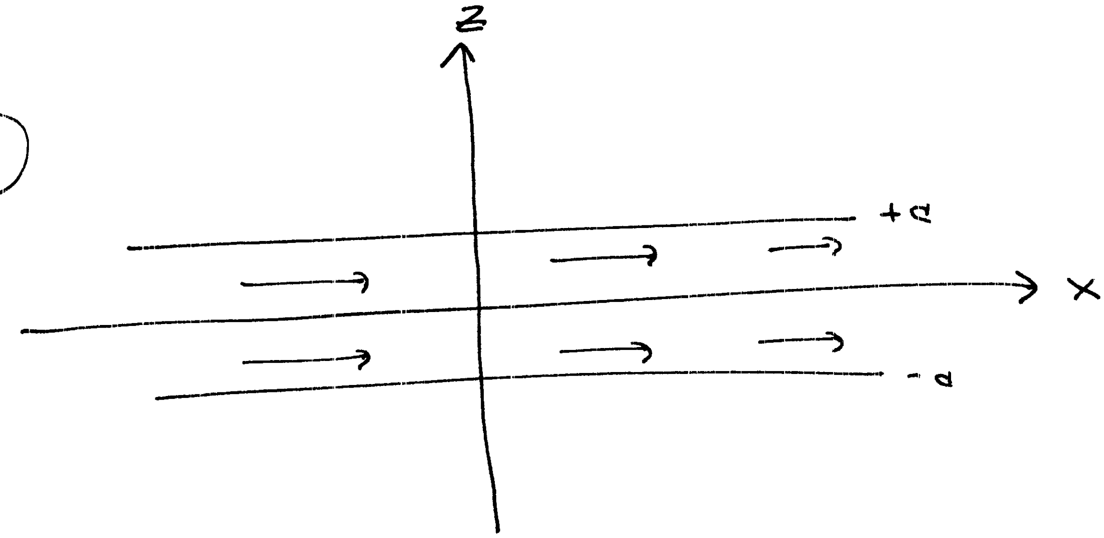
$$= 0.287$$

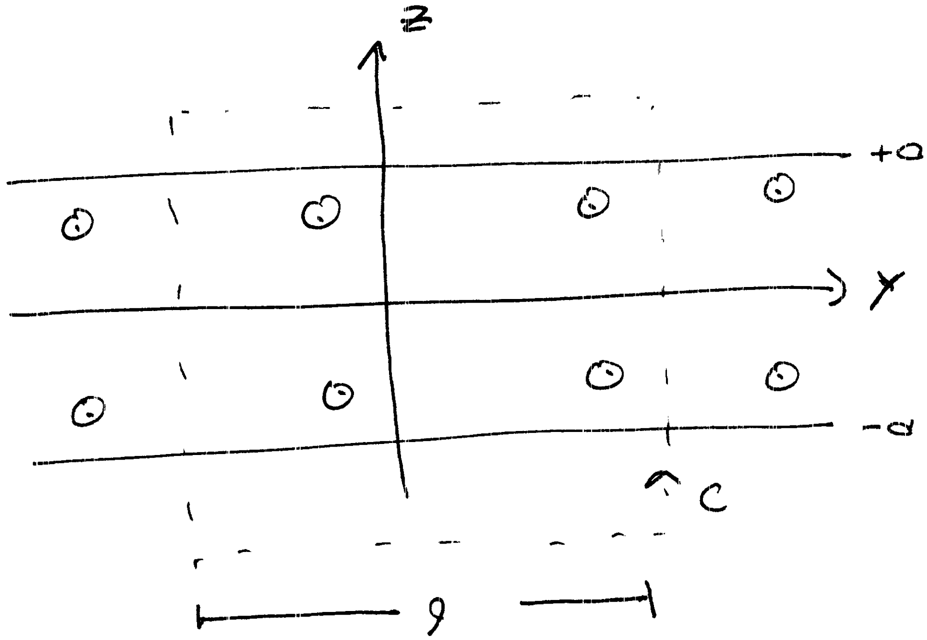
$$B_s = \mu_o K_b \cdot \cos\theta_2 = \mu_o M \cdot 0.287 = 0.37T$$

$$= 6 \cdot (0.061T)$$

Not much of a correction.

(4)





The electric flux through the surface bounded by C

is $\Phi_e = EA = E \cdot 2a \cdot l$

$= 2alE_0 \sin\omega t$

The displacement current is

$$I_d = \varepsilon_0 \frac{d\Phi_e}{dt} = 2\varepsilon_0 \omega a l E_0 \cos\omega t$$

Using the RHR, for a displacement current out of the page in the second figure, the magnetic field above $z > a$, points to the left

$$\vec{B}_t = -B_0 \hat{y}$$

and the field below

$$\vec{B}_b = B_0 \hat{y}$$

<u>Amperes Law</u>

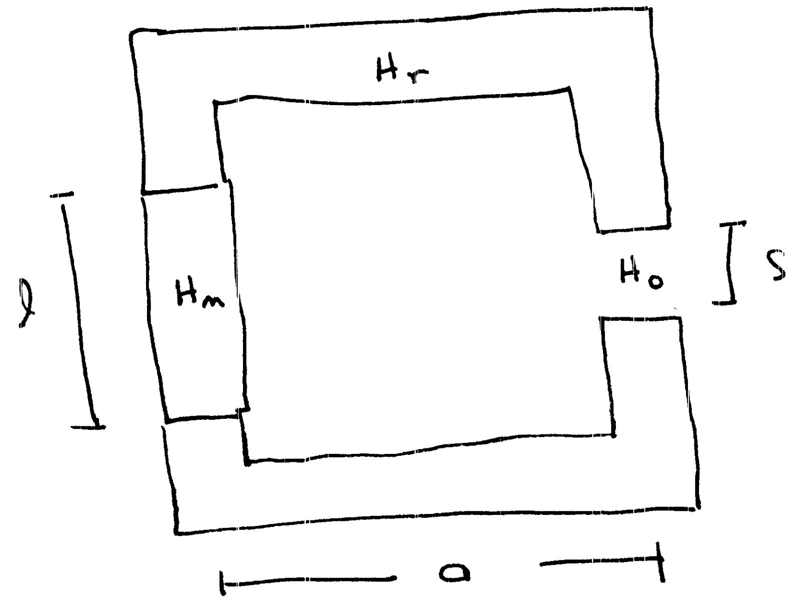
$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 (I_{enc} + I_d) = \mu_0 I_d$$

$$= +B_0 l + B_0 l = \mu_0 I_d$$

$$B_0 = \frac{\mu_0 I_d}{2l} = \mu_0 \epsilon_0 \omega a E_0 \cos \omega t$$

Direction $-\hat{y}$ by RHR wire.

(5)



Ampere's Law

$$\oint \vec{H} \cdot d\vec{l} = I_{fenc} = 0$$

$$H_m l + H_o s + H_r(4a - l - s) = 0$$

Using No Magnetic Monopoles, the magnetic field must be the same at all points around the circle.

$$B = B_o = B_i = B_m$$

Outside $\mu_o H_o = B_o = B$

In the iron $B_r = B = \mu_r \mu_o H_r$

In the magnetic material,

$$H_m = \frac{B}{\mu_0} - M$$

Substitute,

$$\left(\frac{B}{\mu_0} - M\right) l + \frac{B}{\mu_0} s + \frac{B}{\mu_0 \mu_r} \overbrace{(40 - l - s)}^{\gamma} = 0$$

$$Bl - \mu_0 M l + Bs + \frac{B}{\mu_r}\gamma = 0$$

$$B\left(l + s + \frac{\gamma}{\mu_r}\right) = \mu_0 M l$$

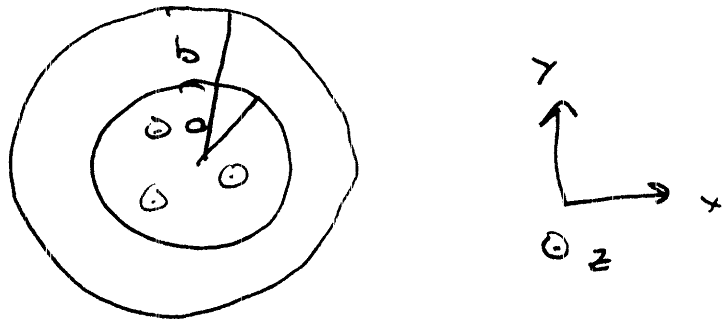
$$\gamma = (4.3\text{cm} - 2\text{cm} - 1\text{cm})$$
$$= 9\text{cm}$$

$$B = \frac{\mu_0 M l}{l + s + \frac{\gamma}{\mu_r}}$$

$$B = \frac{(4\pi \times 10^{-7}\,\text{Tm/A})(1.02 \times 10^{6}\,\text{A/m})(0.02\text{m})}{\left(0.02\text{m} + 0.01\text{m} + \frac{0.09\text{m}}{100}\right)}$$

$$B = \cancel{0.66\,\text{T}}\ 0.83\,\text{T}$$

6



Compute $\vec{H}$

$s < a$

$$I_{fenc} = \int_0^{2\pi} d\phi \int_0^s s\, J\, ds$$

$$= 2\pi \cdot J_0 a \cdot \int_0^s \sin\frac{\pi s}{a}\, ds$$

$$= -2\pi J_0 a \cdot \frac{a}{\pi} \cos\frac{\pi s}{a}\Big|_0^s$$

$$= 2 J_0 a^2 \left(1 - \cos\frac{\pi s}{a}\right)$$

Ampere's Law

$$\int \vec{H} \cdot d\vec{l} = I_{fenc} = 2\pi s\, H$$

RHR

$$\vec{H} = \frac{I_{fenc}}{2\pi s}\hat{\phi} = \frac{2 J_0 a^2}{2\pi s}\left(1 - \cos\frac{\pi s}{a}\right)\hat{\phi}$$

Since there is no material, $\vec{M} = 0$ $s < a$, and

$$\vec{B} = \mu_0 \vec{H} = \frac{J_0 a^2 \mu_0}{\pi s}\left(1 - \cos\frac{\pi s}{a}\right)$$

<u>For $s > a$</u>

$$I_{fenc} = 2 J_0 a^2 \left(1 - \cos\frac{\pi a}{a}\right)$$

$$= 4 J_0 a^2$$

$$\vec{H} = \frac{I_{fenc}}{2\pi s}\hat{\phi} = \frac{J_0 a^2}{\pi s}\hat{\phi}$$

For $s > b$, $\vec{M} = 0$, $\vec{B} = \mu_0 \vec{H} = \frac{J_0 a^2 \mu_0}{\pi s}\hat{\phi}$

For $a < s < b$, $\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$

$$\vec{B} = \frac{\mu_r \mu_0 J_0 a^2}{\pi s}\hat{\phi}$$

$$\vec{M} = \chi_m \vec{H} = (\mu_r - 1)\vec{H}$$

$$= \frac{2(\mu_r - 1) J_0 a^2}{\pi s}\hat{\phi}$$