Vector Operators

Del (nabla) - operator

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Gradient (grad) - scalar $\longrightarrow$ vector

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$\Rightarrow$ Vector that points in the direction in which the function f changes the most rapidly.

Divergence (vector $\rightarrow$ scalar)

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$\Rightarrow$ The rate at which $\vec{A}$ spreads out.

Curl (vector $\rightarrow$ vector)

$$\text{curl } \vec{A} = \nabla \times \vec{A}$$

$$= \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$\Rightarrow$ Rotation of $\vec{A}$.

Laplacian (scalar $\rightarrow$ scalar)

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Divergence and curl must be applied to vector, so $\nabla \cdot f$ and $\nabla \times f$ are nonsense.

- We can define the action of the Laplacian on a vector

$$\nabla^2 \vec{A} = \left(\nabla^2 A_x , \nabla^2 A_y , \nabla^2 A_z \right)$$

Working with vectors

- ∇, $\nabla\cdot$, $\nabla\times$, ∇^2 are operators

- Operators in general do <u>NOT</u> commute, so

  $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ does not imply

  $$\vec{r}\cdot\nabla = \nabla\cdot\vec{r}$$

  $$\vec{r}\cdot\nabla = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

  $$\nabla\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

- In general to prove two operators are equal, $O_1 = O_2$, we show that

  $O_1 f = O_2 f$ for all f.

<u>Operator Identities</u> The rules of one-dimensional calculus can be extended to vector operators.

<u>Product Rule</u> $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

note cannot be derived by applying

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A})$$

because operators do not commute.

$$\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$$

Second derivatives

$$\nabla \times \nabla f = 0$$

$$(\nabla \times \nabla) \cdot \vec{A} = 0$$

$$(\nabla \times \nabla) \times \vec{A} = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$