Vector Potential

Since $\nabla \cdot \vec{B} = 0$, the magnetic field is uniquely determined by the curl of some function, called the vector potential, $\vec{A}$

$$\vec{B} = \nabla \times \vec{A}$$

Try the definition,

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')\, d\tau'}{r''}$$

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\vec{J}(\vec{r}\,')}{r''} \right) d\tau'$$

Vector Identity

$$\nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times \nabla f$$

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int \left[\frac{1}{r''} \nabla \times \vec{J}(\vec{r}\,') - \vec{J} \times \nabla \left(\frac{1}{r''} \right) \right] d\tau'$$

The first term is zero because the curl is taken with respect to unprimed coordinates but $\vec{J}$ depends on primed coordinates.

As before, $\nabla\left(\frac{1}{r''}\right) = -\frac{\hat{r}''}{r''^2}$

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}') \times \hat{r}''}{r''^2}\, d\tau' = \vec{B}$$

We can also write

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l}'}{r''} \quad \text{or} \quad \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\, da'}{r''}$$

Qualitatively $\vec{B}$ points in the direction of the circulation of $\vec{A}$, or if your thumb points in the direction of $\vec{B}$, your fingers curl in the direction of $\vec{A}$.

Consider,

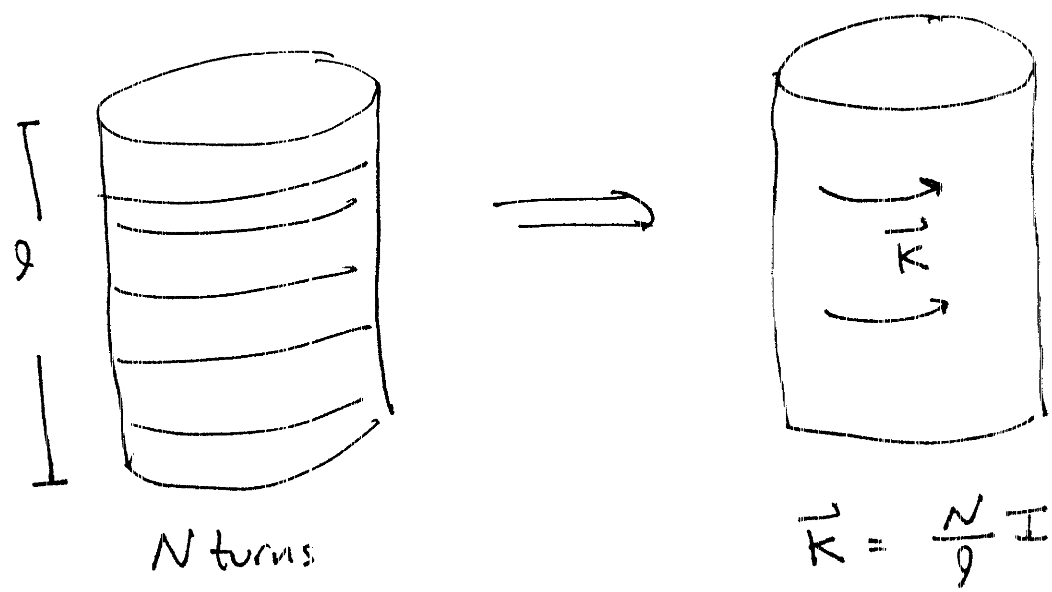
$$\int_S (\nabla \times \vec{A}) \cdot \hat{n}\, da = \int_S \vec{B} \cdot \hat{n}\, da \equiv \Phi_m$$

magnetic flux

$$= \oint_C \vec{A} \cdot d\vec{l}$$

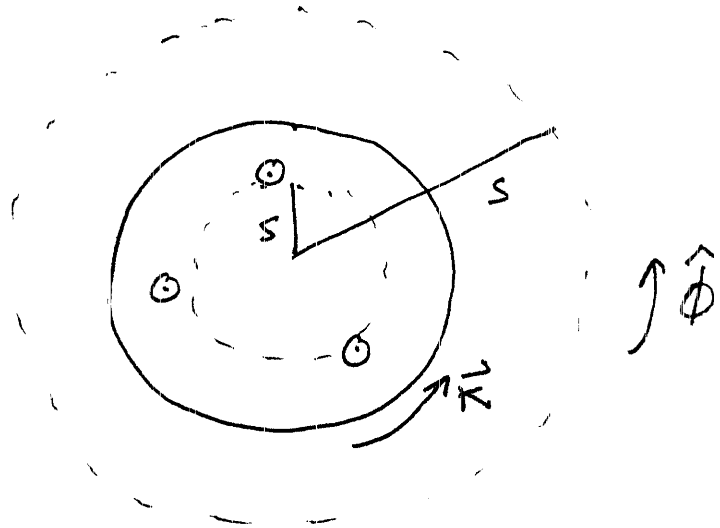
$\Rightarrow$ This implies we can use Ampere's Law like reasoning to find the vector potential.

<u>Ex</u> Vector potential of infinite solenoid carrying current I with N turns per length l.



$$\vec{K} = \frac{N}{l} I \hat{\phi}$$
$$= n I \hat{\phi}$$

End View



$B = \mu_0 n I$

$= \mu_0 K$

By symmetry, and using the idea the $\vec{B}$ points in the direction of the circulation of $\vec{A}$, $\vec{A}$ is circular.

Outside Solenoid

$$\oint_C \vec{A} \cdot d\vec{l} = 2\pi s A = \Phi_m = BA = B\pi a^2$$

$$A = \frac{\pi a^2 B}{2\pi s} = \frac{1}{2}\frac{Ba^2}{s} = \frac{1}{2}\frac{K\mu_0 a^2}{s}$$

$$\vec{A} = \frac{1}{2}\frac{K\mu_0 a^2}{s}\hat{\phi}$$

Inside Solenoid

$$\oint_C \vec{A} \cdot d\vec{l} = 2\pi s A = \Phi_m = B\pi s^2$$

$$\vec{A} = \frac{sB}{2}\hat{\phi} = \frac{s\mu_0 K}{2}\hat{\phi}$$

Check Inside

$$\nabla \times \vec{A} = \frac{1}{s}\left(\frac{\partial (s A_\phi)}{\partial s}\right)\hat{z}$$

$$= \frac{1}{s}\left(\frac{\partial}{\partial s}\frac{s^2 B}{2}\right)\hat{z}$$

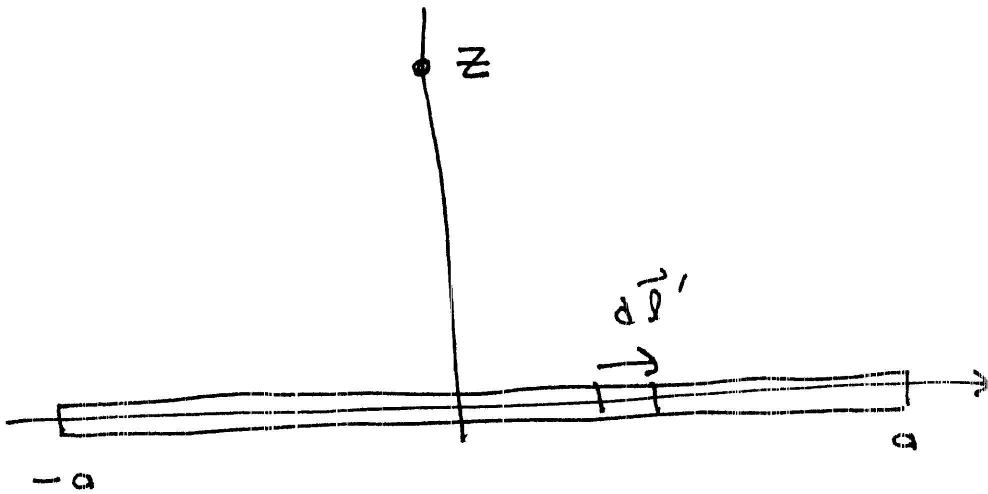
$$= B\hat{z} \quad \checkmark$$

Outside

$$\nabla \times \vec{A} = \frac{1}{s}\left(\frac{\partial \left(s \cdot \frac{1}{2}\frac{Ba^2}{s}\right)}{\partial s}\right)\hat{z}$$

$$= 0 \quad \checkmark$$

Ex Vector potential finite wire along axis



$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_{-a}^{a} \frac{d\vec{l}\,'}{r''}$$

$$d\vec{l} = dx'\,\hat{x}$$

$$\vec{r} = (0,0,z) \qquad \vec{r}\,' = (x',0,0)$$

$$r'' = \sqrt{x'^2 + z^2}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\,\hat{x} \underbrace{\int_{-a}^{a} \frac{dx'}{\sqrt{x'^2+z^2}}}_{2\,\mathrm{arcsinh}\left(\frac{a}{z}\right)} =$$

Newer version of Maple

$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{x}\cdot 2\ln\left[\frac{a+\sqrt{a^2+z^2}}{z}\right]$$

$$\vec{A} = \frac{\mu_0 I \hat{x}}{2\pi}\ln\left[\frac{a+\sqrt{a^2+z^2}}{z}\right]$$

If the wire segment becomes very long,
$\sqrt{a^2+z^2} \sim a$

$$\vec{A} = \frac{\mu_0 I \hat{x}}{2\pi}\ln\left(\frac{2a}{z}\right)$$

$$= -\frac{\mu_0 I}{2\pi}\ln(z)\,\hat{x} \quad + \text{constant}$$

Magnetic Field (cylindrical coordinates)

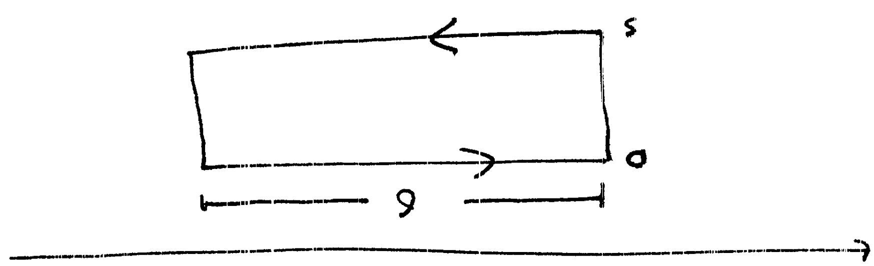
$$\nabla\times\vec{A} = \frac{1}{s}\frac{\partial A_z}{\partial\phi}\hat{s} - \frac{\partial A_z}{\partial s}\hat{\phi}$$

Change the potential to cylindrical coordinates

$$\hat{x} \rightarrow \hat{z} \quad , \quad z \rightarrow s$$

$$\vec{A} = \frac{-\mu_0 I}{2\pi} \ln(s) \ \hat{z}$$

$$\nabla \times \vec{A} = -\frac{\partial A_z}{\partial s} \hat{\phi} = \frac{-\mu_0 I}{2\pi s} \hat{\phi} \quad \checkmark$$



$$\Phi_m = l \int B(s)\, ds = \frac{l \mu_0 I}{2\pi} \int_a^s \frac{ds}{s}$$

$$= \frac{l \mu_0 I}{2\pi} \ln\left(s/a \right) = A_a l - A(s) l$$

$$\vec{A}(s) = \left(A_a - \frac{\mu_0 I}{2\pi} \ln\left(s/a \right) \right) \hat{z}$$

$$\nabla \times \vec{A} = \frac{1}{s} \frac{\partial v_z}{\partial \phi} \hat{s} - \frac{\partial v_z}{\partial s} \hat{\phi}$$

$$= \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

Differential Eqn for $\vec{A}$

$$\nabla \times \vec{B} = \mu_0 \vec{J} = \nabla \times (\nabla \times \vec{A})$$

$$= \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \qquad \text{vector identity}$$

Guage Choices

The electric potential is undefined up to a constant. This is because the field is the measurable quantity and $-\nabla V$ kills the constant. It is more complicated for the vector potential where $\vec{B} = \nabla \times \vec{A}$. Here we can change $\vec{A}$ by adding the gradient of any function $\vec{A}' = \vec{A} + \nabla \lambda$ since

$$\vec{B} = \nabla \times \vec{A}' = \nabla \times \vec{A} + \underbrace{\nabla \times \nabla \lambda}_{0}$$

As such, we can pick λ so $\nabla \cdot \vec{A}' = 0$,

$$\nabla \cdot \vec{A}' = \nabla \cdot A + \nabla^2 \lambda = 0$$

$$\Rightarrow \nabla^2 \lambda = - \nabla \cdot \vec{A}$$

Choose $\nabla \cdot \vec{A} = 0$, called gauge choice. This choice is called the <u>Coulomb Gauge</u>.

In the Coulomb Gauge,

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

which is just 3 copies of Poisson's eqn.

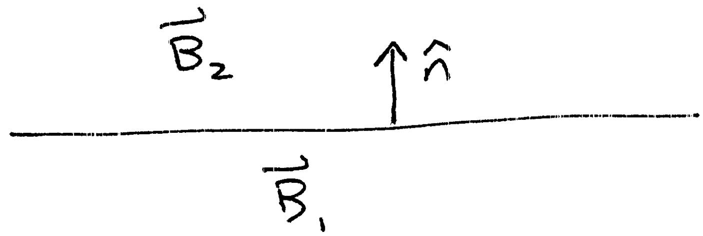
<u>Magnetostatic Scalar Potential</u> -

For regions with $\vec{J} = 0$, one has $\nabla \times \vec{B} = 0$ and can introduce $\vec{B} = -\nabla V_m$ where V_m is the magnetostatic scalar potential.

Since $\nabla \cdot \vec{B} = 0$, $\nabla^2 V_m = 0$ and we have Laplace's eqn to solve again.

Boundary Conditions

From before, $\vec{B}_2 - \vec{B}_1 = \mu_0 (\vec{K} \times \hat{n})$



Since $\vec{B} = \nabla \times \vec{A}$, $\vec{A}$ must be continuous at an interface $\vec{A}_2 = \vec{A}_1$

The normal of $\vec{A}$ is discontinuous by the current

$$\frac{\partial \vec{A}_2}{\partial n} - \frac{\partial \vec{A}_1}{\partial n} = \mu_0 \vec{K}$$

- Since $\vec{A}$ continuous, $\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}$ are continuous along surface $\hat{n} = \hat{z}$.
- This only leaves taking the curl and fixing up z components.

For the magnetostatic scalar potential, we are left applying $\vec{B}_2 - \vec{B}_1 = \mu_0 (\vec{K} \times \hat{n})$ at the surface.

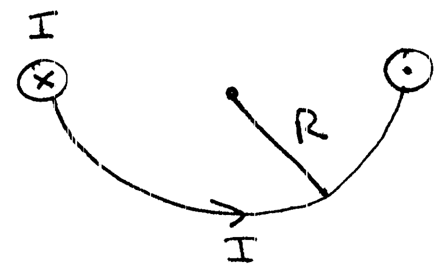
<u>Ex</u> Vector Potential of Spinning Sphere with uniform surface current density σ and angular velocity ω.

$$\vec{A}(\vec{r}, \theta, \phi) = \begin{cases} \dfrac{\mu_0 R \omega \sigma}{3} \, r \sin\theta \, \hat{\phi} & r \leq R \\[2ex] \dfrac{\mu_0 R^4 \omega \sigma}{3} \, \dfrac{\sin\theta}{r^2} \, \hat{\phi} & r \geq R \end{cases}$$

The text attacks this problem by direct integration of the current density $\vec{J}$ through

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,')}{r''} \, da'$$

Ex Compute Vector Potential at Center of Circular Segment



Sln
$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}\,'}{r''}$$

$$d\vec{l}\,' = R\,d\phi'\,\hat{\phi}'$$

$$= \frac{\mu_0 I}{4\pi}\int_{\pi}^{2\pi} d\phi'\,\hat{\phi}'$$

$$= \frac{\mu_0 I}{4\pi}\int_{\pi}^{2\pi} d\phi'\,\left(-\sin\phi'\,\hat{x} + \cos\phi'\,\hat{y}\right)$$

$$= \frac{\mu_0 I}{4\pi}\cos\phi'\,\hat{x}\,\Big|_{\pi}^{2\pi}$$

$$= \frac{\mu_0 I}{2\pi}\,\hat{x}$$