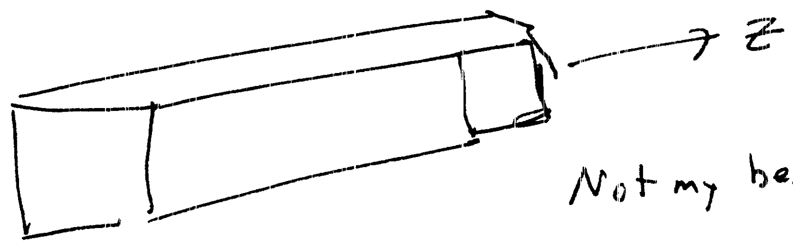
Wave Guides

If we let EM waves travel in a conducting channel, we now must meet boundary conditions at the channel surface.



Not my best picture.

If the channel is a perfect conductor, $\vec{E}=0$ in the material. By Faraday's Law, $\partial\vec{B}/\partial t = 0$ in the conductor. so we may choose $\vec{B}=0$

The wave must then satisfy

$$\vec{E}^{\parallel} = 0 \qquad \text{and} \qquad B^{\perp} = 0$$

at the guide.

If we restrict ourselves to waves of a single frequency that propagate in the +z direction, our wave solutions are

$$\vec{E} = \vec{E}_0(x,y)\, e^{i(kz-\omega t)}$$

$$\vec{B} = \vec{B}_0(x,y)\, e^{i(kz-\omega t)}$$

To satisfy Maxwell's eqns and the boundary conditions we will need a component of the wave in the direction of propagation.

$$\vec{E}_0 = E_{0x}\,\hat{x} + E_{0y}\,\hat{y} + E_{0z}\,\hat{z}$$

$$\vec{B}_0 = B_{0x}\,\hat{x} + B_{0y}\,\hat{y} + B_{0z}\,\hat{z}$$

Substitution of the wave solution into Maxwell's eqns yields the following conditions on E_z, B_z.

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] E_z = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] B_z = 0$$

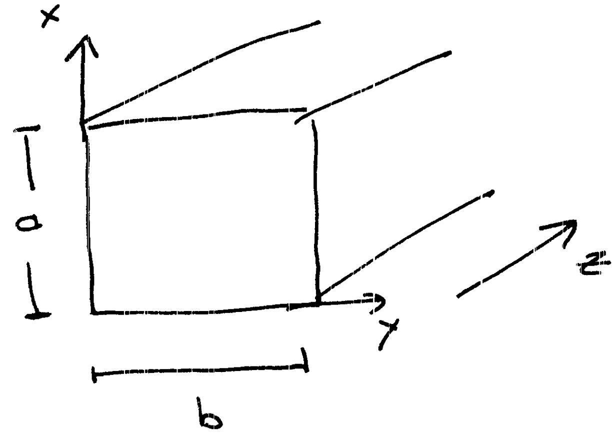
Transverse Electric (TE) Waves - $E_z = 0$

Transverse Magnetic (TM) Waves - $B_z = 0$

TEM Waves - $E_z = B_z = 0$

$\Rightarrow$ Cannot exist in hollow wave guide.

Rectangular Wave Guides



TE Mode ($E_z = 0$)

Separation of Variables

$$B_z = X(x)\,Y(y)$$

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} + \left(\left(\frac{\omega}{c}\right)^2 - k^2\right) X Y = 0$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$$

Introduce separation constants

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -k_x^2 \qquad \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -k_y^2$$

$$-k_x^2 - k_y^2 + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$$

Solution

$$X(x) = A \sin k_x x + B \cos k_x x$$

Boundary Condition

$$B^{\perp} = 0 \implies B_x = 0$$

From Maxwell's Eqns, $\dfrac{\partial B_z}{\partial x} = \dfrac{\partial X}{\partial x} = 0$ at $x = 0$, $x = a$

$$\implies A = 0$$

$$\implies k_x = \frac{m\pi}{a} \implies X(x) = B \cos \frac{m\pi x}{a}$$

Likewise

$$k_y = \frac{n\pi}{b} \qquad Y(y) = B \cos \frac{n\pi y}{b}$$

Solution

$$B_z(x,y) = B_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

TE_{mn} mode

– Assume $a \geq b$ by convention.

Wave number

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2 - k_y^2}$$

$$= \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left(\frac{m}{a}\right)^2 - \pi^2\left(\frac{n}{b}\right)^2}$$

For the wave to travel without attenuation, k must be real. Let the lowest frequency for attenuation be ω_{mn}, called the cutoff frequency.

$$0 = \sqrt{\left(\frac{\omega_{mn}}{c}\right)^2 - k_x^2 - k_y^2}$$

$$\omega_{mn} = c\sqrt{k_x^2 + k_y^2} = c\pi\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right)^{1/2}$$

The lowest cutoff frequency ω_{10} is the lowest frequency any wave propagates without attenuation

$$\omega_{10} = \frac{c\pi}{a}$$

Dispersion Relation

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

Phase Velocity

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}} \geq c$$

since $\omega \geq \omega_{mn}$.

Group Velocity (wave packet velocity)

$$v_g = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c$$

$\Rightarrow$ Energy travels less than c.

$\Longrightarrow$ Coaxial Transmission line can support TEM modes.