Ampere's Law (Magnetostatic)

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int_S \vec{J} \cdot d\vec{a}$$

$\parallel$ Stokes

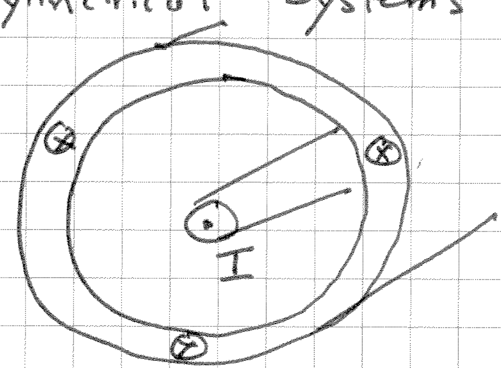
$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$I_{enc} \equiv \int_S \vec{J} \cdot d\vec{a}$$

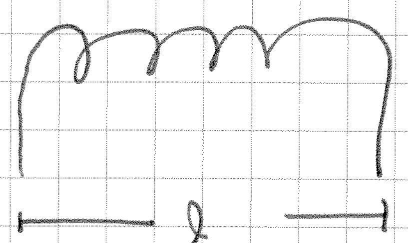
The integral of the magnetic field around the closed curve C is proportional to the current flowing ~~the~~ a surface S bounded by C.

Ampere's Law is true for all magnetostatic systems. It is a useful calculation tool for the magnetic field only for systems with high symmetry:

1) Cylindrical Systems

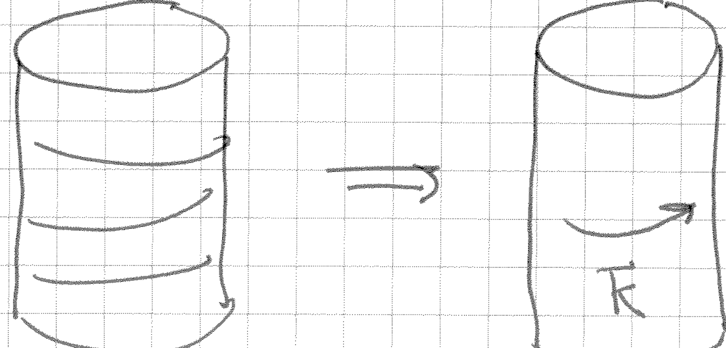


2) Solenoids in the infinite solenoid approximation



$B_i = \mu_0 \frac{N}{l} I$ inside

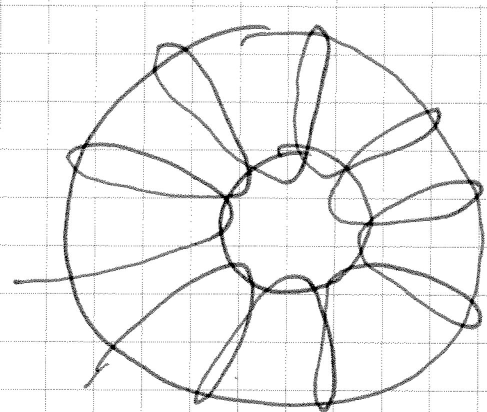
$B_o = 0$ outside



$|\vec{K}| = \frac{N}{l} I$

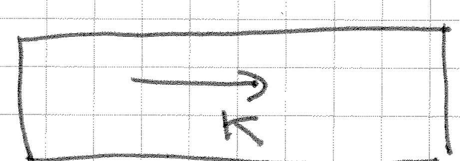
$B_i = \mu_0 K$

3) Toroidal Solenoids

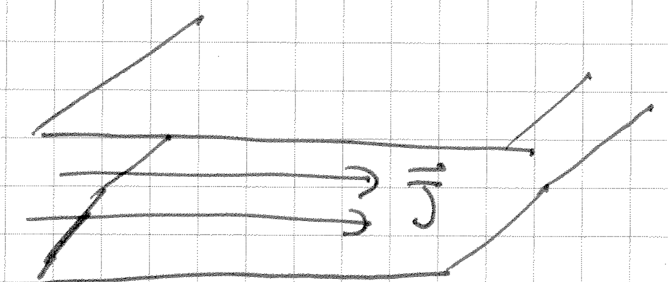


4) Planar Currents

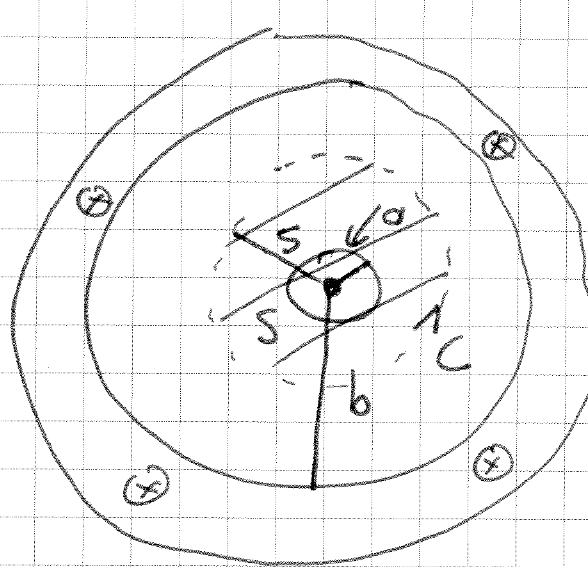
Sheets



Infinite Volume Currents



Ex Co-axial cable, inner radius a outer radius b. Carries current I out of page in central conductor and into page in outer conductor.



By right-hand rule for wire, magnetic field is counter-clockwise between conductors.

⇒ Convention for positive surface normal - To select the positive normal for surface S bounded by C curl fingers in the direction of C, thumb points in the direction of $+\hat{n}$ (right hand).

⇒ For C above, $\hat{n}$ out of page.

Between the conductors, $a < s < b$,

$$I_{enc} = +I = \vec{I} \cdot \hat{n}$$

The positive normal points out of the page as does $\hat{n}$.

At all points along C, $\vec{B} \parallel d\vec{l}$ and of the same magnitude so

$$\oint \vec{B} \cdot d\vec{l} = \underbrace{2\pi r}_{\text{Length of } C} B = \mu_0 I_{enc}$$

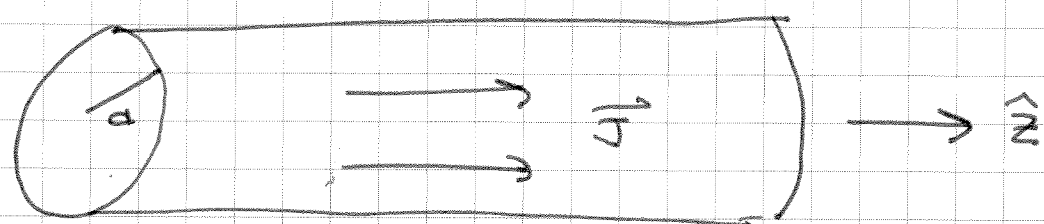
$$\vec{B} = \frac{\mu_0 I}{2\pi r} \quad \text{CCW}$$

<u>Outside both conductors</u>

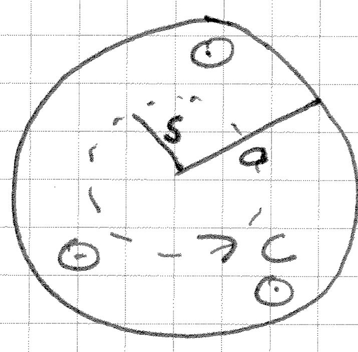
$$I_{enc} = \vec{I}_{wire} \cdot \hat{n} + (-\vec{I}_{wire}) \cdot \hat{n} = 0$$

$$\vec{B} = 0$$

Ex A cylindrical region contains a non-uniform current density $\vec{J} = \gamma s \hat{z}$ where γ constant. The current is confined to $s < a$.



End View



By RHR, positive normal out of page

Region I $s < a$

$$I_{enc} = \int_S \vec{J} \cdot d\vec{a} = \int_S \vec{J} \cdot \hat{z} \, da$$

$$= \int_S \gamma s \, da$$

$$da = ds\, s\, d\phi$$

$$I_{enc} = \int \gamma s\, da = \int_0^{2\pi}\int_0^{s} \gamma s\,(s\, ds\, d\phi)$$

$$= \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{s} \underbrace{ds\, \gamma s^2}_{\gamma s^3/3}$$

$$= \frac{2}{3}\pi \gamma s^3$$

Ampere's Law

$$\oint_c \vec{B}\cdot d\vec{l} = 2\pi s B = \mu_0 I_{enc}$$

$$\vec{B}_I = \frac{\mu_0 I_{enc}}{2\pi s} \text{ ccw}$$

$$= \frac{\mu_0 \,{}^2/_3\, \pi \gamma s^3}{2\pi s} \text{ ccw}$$

$$= \frac{\mu_0 \gamma s^2}{3} \text{ ccw}$$

$$\vec{B}_I = \frac{\mu_0 \gamma s^2}{3}\hat{\phi}$$

Region II $\quad s > a$

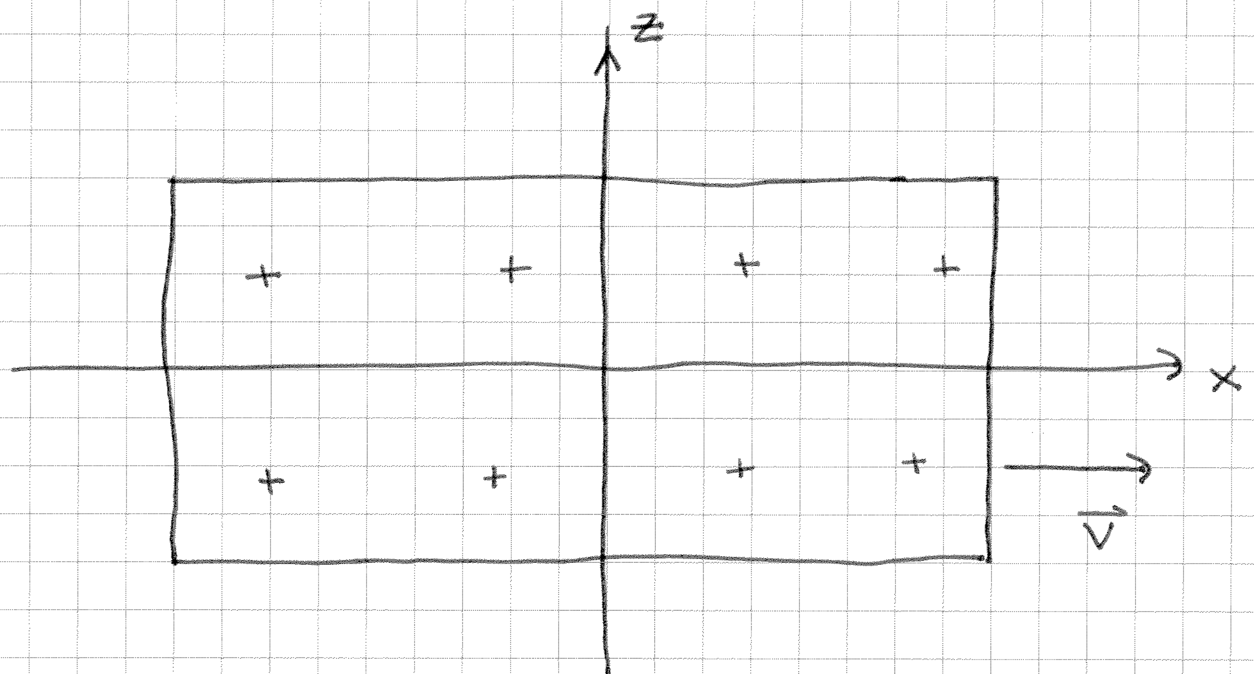
$$I_{enc} = \frac{2}{3}\pi \gamma a^3$$

$$\vec{B}_{II} = \frac{\mu_0 I_{enc}}{2\pi s}\hat{\phi}$$

$$= \frac{\mu_0 \, 2/3 \, \pi \gamma a^3}{2\pi s}$$

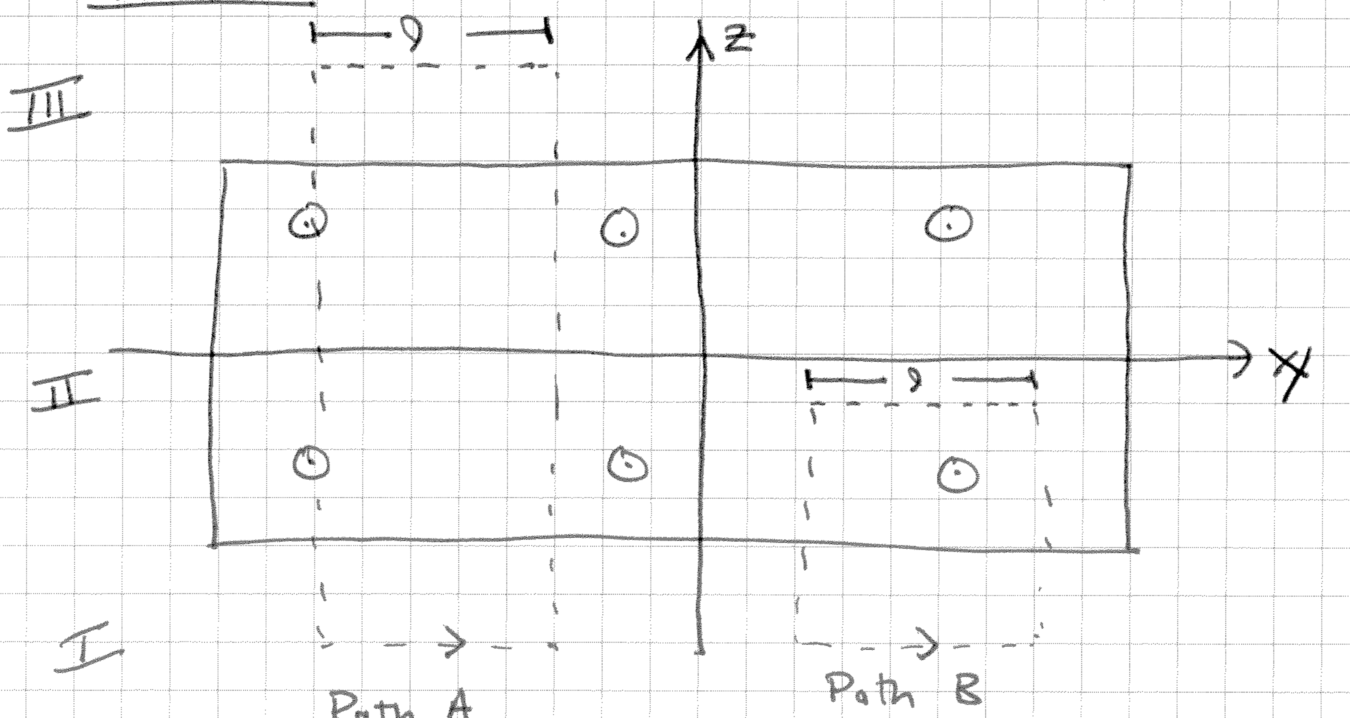
$$= \frac{\mu_0 \gamma a^3}{3s}\hat{\phi} = \frac{\mu_0 \gamma a^3}{3s} \text{ CCW}$$

Ex A slab with uniform volume charge density ρ occupies the region $-a < z < a$. The slab moves with velocity $\vec{v} = v_0 \hat{x}$. Compute magnetic field.



$$\vec{J} = \rho \vec{v} = \rho v_0 \hat{x}$$

End View



Because it is constructed using a cross-product $\vec{B}$ is a psuedo-vector that changes sign under reflection, or just check $d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$

Field points in $-y$ direction above slab, $+y$ direction below slab.

Path A $\quad d\vec{l}_{III} = -dy\,\hat{y} \qquad d\vec{l}_{I} = dy\,\hat{y}$

$$\oint_C \vec{B} \cdot d\vec{l} = \vec{B}_{III} \cdot (-\hat{y})\, l + \vec{B}_{I} \cdot \hat{y}\, l$$

$$= -B_{III}\, l + B_{I}\, l \qquad = \mu_0 I_{enc}$$

By rotation symmetry, $\vec{B}_{I} = -\vec{B}_{III}$

$$-2 B_{III}\, l = \mu_0 I_{enc}$$

$$I_{enc} = 2 a l |\vec{J}| = 2 a l \rho v_0$$

$$-2 B_{III}\, l = \mu_0 2 a l \rho v_0$$

$$\vec{B}_{III} = -\mu_0 a \rho v_0 \hat{y}$$

$$\vec{B}_{I} = -\vec{B}_{III} = \mu_0 \rho v_0 a \hat{y}$$

Path B Let z be location of top of path

$$\oint_{C_B} \vec{B} \cdot d\vec{l} = B_I l - B_{II} l = \mu_0 I_{enc}$$

$$I_{enc} = |\vec{J}| \, l \, (z + a)$$

$$= \rho v_0 l (z + a)$$

$$B_{II} = B_I - \frac{\mu_0 I_{enc}}{l}$$

$$= \mu_0 \rho v_0 a - \mu_0 \rho v_0 (z + a)$$

$$= -\mu_0 \rho v_0 z$$

$$\vec{B}_{II} = -\mu_0 \rho v_0 z \hat{y}$$