Boundary Value Problem with Dielectrics

A linear dielectric $\vec{D} = \varepsilon \vec{E}$ is curl free except at the interfaces. So away from the interfaces, the potential satisfies Laplace's eqn with the boundary conditions:

<u>Continuous</u> $V_i = V_o$

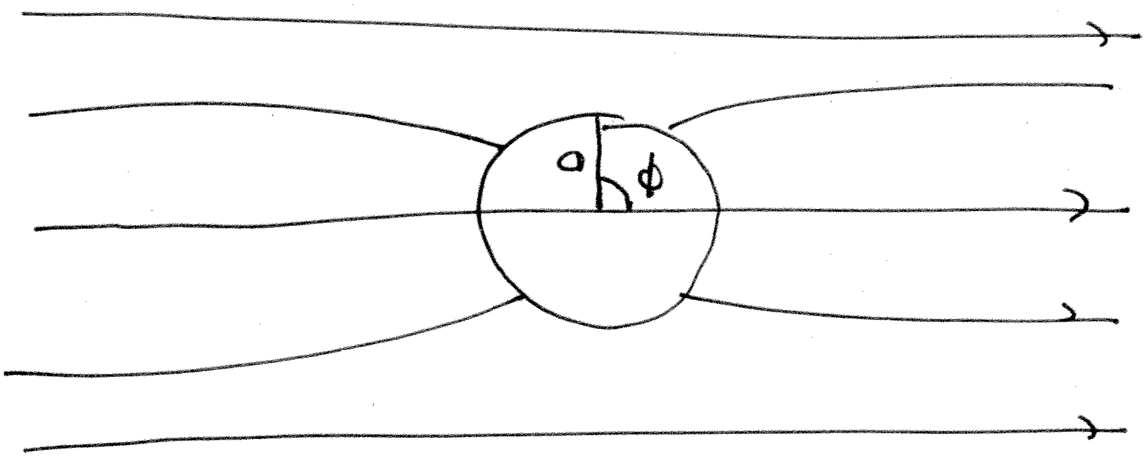
<u>Electrostatic</u> $D_o^{\perp} - D_i^{\perp} = \sigma_f$

which becomes

$$\kappa_o \left.\frac{\partial V_o}{\partial n}\right|_{\text{interface}} - \kappa_i \left.\frac{\partial V_i}{\partial n}\right|_{\text{interface}} = \frac{-\sigma_f}{\varepsilon_o}$$

<u>Ex</u> Dielectric Cylinder in Uniform Field

$$\vec{E}(\pm\infty) = E_0 \hat{x}$$



As $s \to \infty$, $V \to -E_0 x = -E_0 s\cos\phi$

<u>Potential Inside</u> $(s<a)$

$$V_i = \sum_n A_n s^n \cos n\phi + B_n s^n \sin n\phi$$

<u>Potential Outside</u>

$$V_o = \sum_n C_n s^{-n}\cos n\phi + D_n s^{-n}\sin n\phi$$

$$+ E_n s^n \cos n\phi + F_n s^n \sin n\phi$$

As $s \to \infty$, $V_o(s) \to -E_0 s\cos\phi$

$$\Rightarrow F_n = 0, \quad E_n = 0 \quad n=0,\ n \geq 2$$

$$\boxed{E_1 = -E_0}$$

$$V_o = \sum_n A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi$$
$$- E_0 s \cos\phi$$

Apply B.C. at surface of cylinder

$$V_i(a,\phi) = V_o(a,\phi) \qquad \underline{\text{Continuity}}$$

$$V_i(a,\phi) = \sum_n A_n a^n \cos n\phi + B_n a^n \sin n\phi$$

$$= V_o(a,\phi)$$

$$= \sum_n C_n a^{-n} \cos n\phi + D_n a^{-n} \sin n\phi$$
$$\cancel{E_0 a \cos n\phi} \; - E_0 a \cos\phi$$

Use orthogonality to match like terms

$n=0,\ n \geq 2 \qquad A_n a^n = C_n a^{-n}$

$n=1 \qquad A_1 a = C_1/a - E_0 a$

All $n \qquad B_n a^n = D_n a^{-n}$

Electrostatic B.C.

$$\kappa_0 \frac{\partial V_0}{\partial s}\bigg|_a - \kappa_i \frac{\partial V_i}{\partial s}\bigg|_a = \frac{-\sigma_f}{\varepsilon_0} = 0$$

$$\frac{\partial V_0}{\partial s}\bigg|_a = \sum_n -n A_n a^{-(n+1)} \cos n\phi$$
$$-n B_n a^{-(n+1)} \sin n\phi - E_0 \cos\phi$$

$$\frac{\partial V_i}{\partial s}\bigg|_a = \sum_n n A_n a^{n-1} \cos n\phi$$
$$+ n B_n a^{n-1} \sin n\phi$$

Apply BC $(\kappa_0 = 1)$ $(\kappa_i = \kappa)$

$$\frac{\partial V_0}{\partial s}\bigg|_a = \sum_n -n C_n a^{-(n+1)} \cos n\phi$$
$$-n D_n a^{-(n+1)} \sin n\phi - E_0 \cos\phi$$

$$-\kappa \frac{\partial V_i}{\partial s}\bigg|_a \quad -\left[\sum_n \left(n \kappa A_n a^{n-1} \cos n\phi\right.\right.$$
$$\left.\left. n\kappa B_n a^{n-1} \sin n\phi\right) \cancel{E_0 \cos\phi}\right]$$

$$= 0 \quad (\sigma_f = 0)$$

TOPS 35502

All n $(\sin\phi)$

$$-n D_n a^{-(n+1)} - n \kappa B_n a^{n-1} = 0$$

$$D_n = -\kappa B_n a^{2n}$$

$n=0,\ n\geq 2$

$$-n C_n a^{-(n+1)} - n\kappa A_n a^{n-1} = 0$$

$$C_n = -\kappa A_n a^{2n}$$

$n=1$

$$-(1) C_1 a^{-(1+1)} - E_0$$
$$-(1)\kappa A_1 a^{1-1} = 0$$

$$-C_1/a^2 - E_0 - \kappa A_1 = 0$$

Solve BC

All n $(\sin\phi)$

$$B_n a^n = D_n a^{-n}$$

$$D_n = -\kappa B_n a^{2n}$$

$$\Longrightarrow \cancel{A}\ B_n = D_n = 0$$

Likewise $n=0,\ n\geq 2$

$$A_n = C_n = 0$$

$\underline{n=1}$

$$-\frac{C_1}{a^2} - E_0 - \kappa A_1 = 0$$

$$A_1 a = \frac{C_1}{a} - E_0 a$$

$$\frac{C_1}{a^2} - A_1 - E_0 = 0$$

$\underline{\text{Add}}$

$$-E_0 - \kappa A_1 - A_1 - E_0 = 0$$

$$-(\kappa+1)A_1 = 2E_0$$

$$A_1 = \frac{-2E_0}{\kappa+1}$$

$$C_1 = a^2(A_1 + E_0)$$

$$= a^2\left(\frac{-2E_0}{\kappa+1} + E_0\right)$$

$$= a^2 E_0\left(\frac{-2}{\kappa+1} + \frac{\kappa+1}{\kappa+1}\right)$$

$$= +a^2 E_0\left(\frac{\kappa-1}{\kappa+1}\right)$$

Final Solution

Inside

$$V(s,\phi) = \frac{-2\kappa E_0}{1+\kappa} s\cos\phi$$

$$= -\frac{2\kappa E_0}{1+\kappa} x$$

Field

$$\vec{E}_i = -\nabla V_i = \frac{2\kappa E_0}{1+\kappa}\hat{x}$$

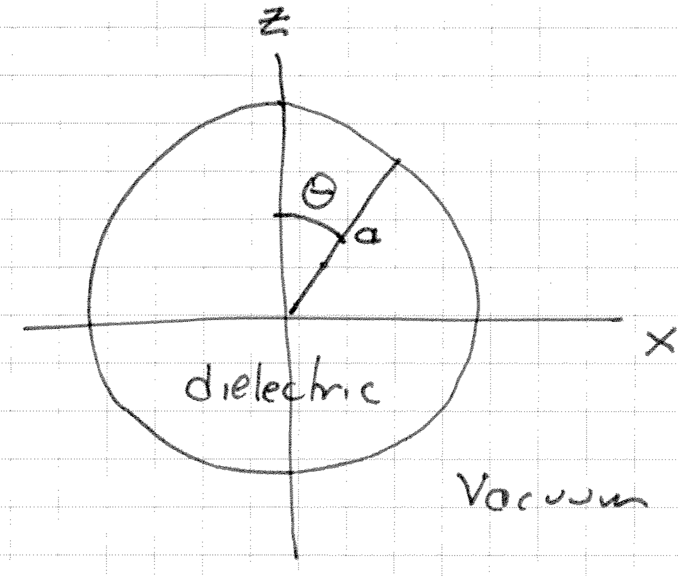
Outside

$$V_o = \frac{E_0 a^2}{s}\left(\frac{\kappa-1}{\kappa+1}\right)\cos\phi - E_0 s\cos\phi$$

<u>From Griffiths</u>

<u>Ex</u> Dielectric sphere $\kappa = \varepsilon_r$ in uniform external field $\vec{E} = E_0 \hat{z}$

$\Longrightarrow$ $V(r) \to -E_0 z$ as $r \to \infty, \theta = 0$



Only actual charge is bound charge at surface, so the potential satisfiest Laplace's eqn inside and outside sphere.

$$V_i = \sum r^n P_n(\cos\theta) A_n$$

$$V_o = \sum B_n r^{-(n+1)} P_n(\cos\theta) - E_o \underbrace{r\cos\theta}_{z}$$

explosive term to satisfy BC at ∞

<u>V is continuous</u>

$$V_i(a,\theta) = V_o(a,\theta)$$

$$\sum_n A_n a^n P_n(\cos\theta) = \sum B_n a^{-(n+1)} P_n(\cos\theta) - E_o a \underbrace{\cos\theta}_{P_1}$$

<u>Term by term</u>

<u>$n \neq 1$</u> $\quad A_n a^n = B_n a^{-(n+1)}$

$$B_n = A_n a^{2n+1}$$

<u>$n=1$</u> $\quad A_1 a = \dfrac{B_1}{a^2} - E_o a$

Electrostatic B.C.

$$\varepsilon_{r,out} \left.\frac{\partial V_o}{\partial r}\right|_a - \varepsilon_{r,in} \left.\frac{\partial V_i}{\partial r}\right|_a = -\sigma_f$$

$\varepsilon_{r,out} = 1 \qquad \sigma_f = 0$

$$\left.\frac{\partial V_o}{\partial r}\right|_a = \varepsilon_r \left.\frac{\partial V_i}{\partial r}\right|_a$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a = \sum_n -(n+1) B_n a^{-(n+2)} P_n(\cos\theta) \quad - E_0 P_1$$

$$\left.\frac{\partial V_i}{\partial r}\right|_a = \sum_n A_n n a^{n-1} P_n(\cos\theta)$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \varepsilon_r \left.\frac{\partial V_i}{\partial r}\right|_a = \sum_n P_n(\cos\theta) \left[-(n+1) B_n a^{-(n+2)} - \varepsilon_r A_n n a^{n-1} \right] - E_0 P_1$$

$$= \sum_n P_n(\cos\theta) \left[-(n+1) a^{n-1} - \varepsilon_r n a^{n-1} \right] A_n - E_0 P_1$$

for $n \neq 1$

By orthogonality, $A_n \quad n \neq 1 \quad = 0$

For $n = 1$,

$$-(1+1)\, B_1 a^{-3} - \varepsilon_r A_1 (1) a^0 - E_0 = 0$$

and from earlier

$$A_1 a = \frac{B_1}{a^2} - E_0 a$$

$$A_1 = \frac{B_1}{a^3} - E_0$$

$$-\frac{2B_1}{a^3} - \varepsilon_r \left(\frac{B_1}{a^3} - E_0 \right) - E_0 = 0$$

$$-\frac{B_1}{a^3} (\varepsilon_r + 2) + \varepsilon_r E_0 - E_0 = 0$$

$$B_1 = a_3 E_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right)$$

$$A_1 = \frac{B_1}{a^3} - E_0 = E_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} - 1 \right)$$

$$= \frac{-3}{\varepsilon_r + 2} E_0$$

So the full potential,

Inside

$$V_i = \cancel{A_1} \; A_1 r P_1(\cos\theta) = A_1 r \cos\theta$$

$$= A_1 z = -\frac{3z}{\varepsilon_r + 2} E_o$$

Field

$$\vec{E}_i = -\nabla V = \frac{3}{\varepsilon_r + 2} E_o \hat{z}$$

Outside

$$V_o = B_1 r^{-2} P_1(\cos\theta) - E_o r \cos\theta$$

$$= \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) \frac{a^3}{r^2} E_o \cos\theta - E_o z$$

Note -- Dipole field.