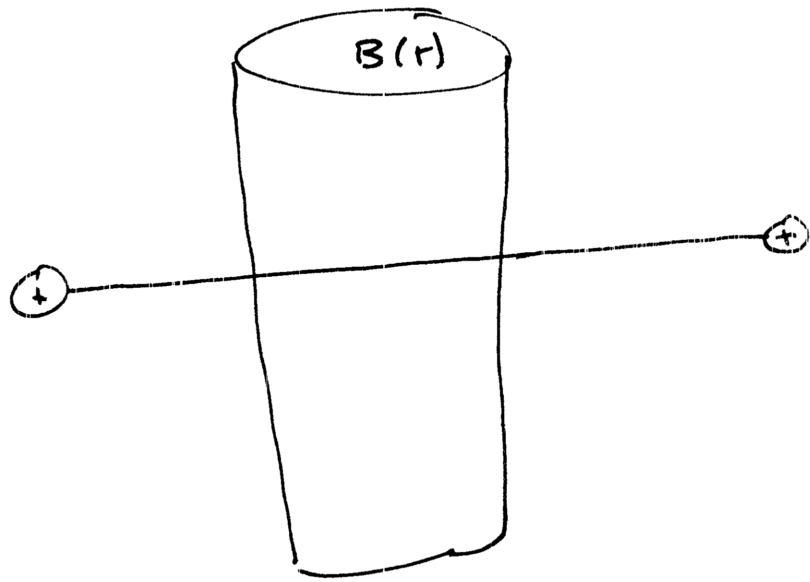
Angular Momentum

We have already encountered a situation where it was evident that angular momentum must reside in the field.



As the magnetic field is turned off, the charges begin to rotate. For conservation of angular momentum, angular momentum must exist in the field and be transferred to the charges.

The momentum density is

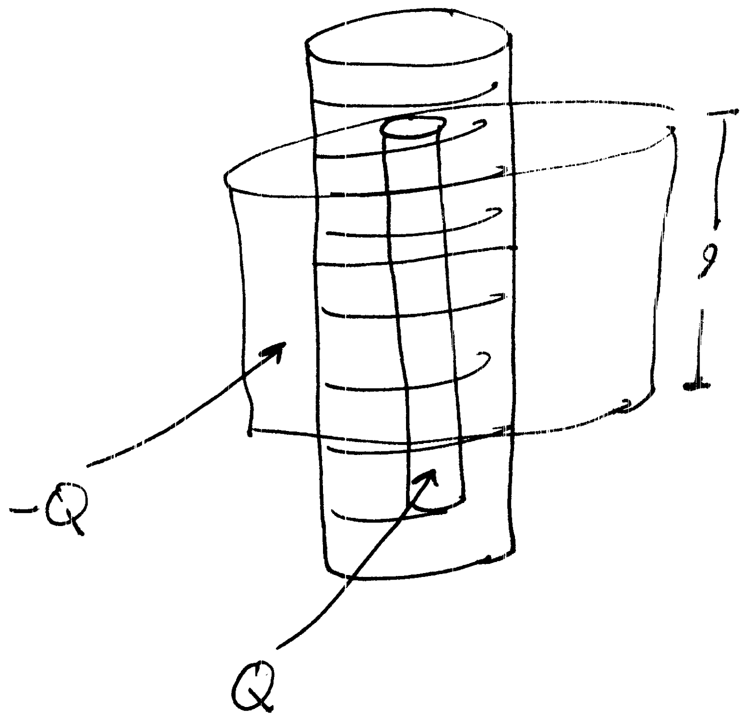
$$\vec{\mathcal{P}}_{em} = \mu_0 \varepsilon_0 \vec{S}$$

The angular momentum density must then be

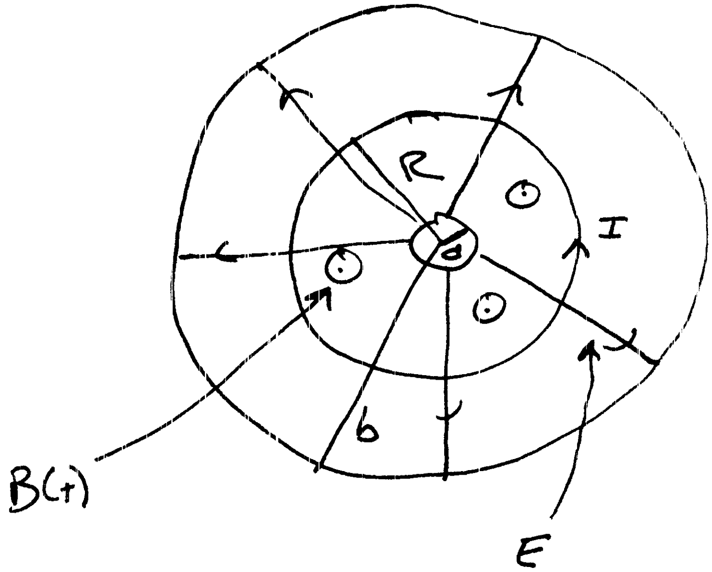
$$\vec{\ell}_{em} = \vec{r} \times \vec{\wp}_{em} = \mu_0 \varepsilon_0 \vec{r} \times \vec{S}$$

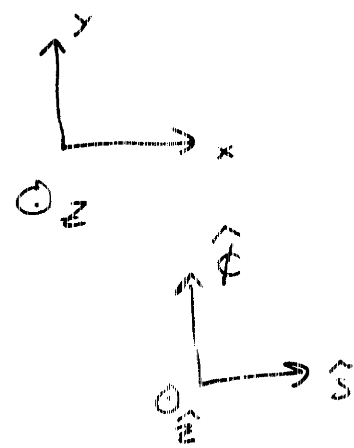
$$= \varepsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$$

Griffith's 8.4 Very similar to the two charge problem. Solenoid carries current $I(t)$. Inside the solenoid is a cylinder of length l and charge Q outside the solenoid is a cylinder of length l and charge $-Q$. Assume everything is quasi-infinite



End View





The electric field produced by $\pm Q$ is static

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s} = \frac{Q}{2\pi\epsilon_0 l s}\hat{s}$$

The magnetic field in the solenoid is as always

$$\vec{B}(t) = \mu_0 n I(t)\hat{z}$$

The ~~total angular~~ momentum density in the solenoid

is

$$\vec{\wp}_{em} = \mu_0 \epsilon_0 \vec{S}$$

$$= \cancel{\mu_0}\, \epsilon_0 \vec{E}\times\vec{B}$$

$$= \epsilon_0 \left(\frac{Q}{2\pi\epsilon_0 l s}\hat{s}\right)\times\left(\mu_0 n I(t)\hat{z}\right)$$

$$\vec{p}_{em} = -\frac{\mu_0 n Q I}{2\pi l s}\,\hat{\phi}$$

The angular momentum density about the z-axis is

$$\vec{l}_{em} = \vec{s} \times \vec{p}_{em} = s\,\hat{s} \times \vec{p}_{em}$$

$$= -\frac{\mu_0 n Q I}{2\pi l}\,\hat{z}$$

The total angular momentum in the length l is this multiplied by the volume

$$\vec{L} = \vec{l}_{em} \cdot \pi(R^2 - a^2)\,l$$

$$= \frac{-\mu_0 n Q I}{2}\,(R^2 - a^2)\,\hat{z}$$

This angular momentum is transferred to the cylinders as the field is turned off

Using an Amperian path of radius s, Faraday's law becomes $(s < R)$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a}$$

$$2\pi s E = -\frac{d}{dt}\left(\pi s^2 B\right) = -\frac{d}{dt}\left(\pi s^2 \mu_0 n I\right)$$

$$= -\pi s^2 \mu_0 n \frac{dI}{dt}$$

$$\vec{E}_i = -\frac{1}{2} s \mu_0 n \frac{dI}{dt} \hat{\phi} \qquad s < R$$

If $s > R$,

$$2\pi s E = -\frac{d}{dt}\left(\pi R^2 B\right)$$

$$\vec{E}_o = -\frac{1}{2} \frac{R^2}{s} \mu_0 n \frac{dI}{dt} \hat{\phi}$$

These fields exert a torque on the cylinders

For the outer cylinder,

$$\vec{\Gamma}_o = b\hat{s} \times \left(-Q\vec{E}_o(b)\right) =$$

$$= b\left(\frac{1}{2} Q \frac{R^2}{b} \mu_0 n \frac{dI}{dt}\right) \hat{s} \times \hat{\phi}$$

$$\vec{T}_o = \frac{1}{2} Q R^2 \mu_o n \frac{dI}{dt} \hat{z}$$

The total angular momentum is the integral of the torque

$$\Delta \vec{L}_o = \int \vec{T}_o \, dt$$

$$= \frac{1}{2} Q R^2 \mu_o n \hat{z} \int_{I_o}^{0} \frac{dI}{dt} \, dt$$

$$= -\frac{1}{2} Q R^2 \mu_o n I_o \hat{z}$$

where I_o is the initial current.

The torque on the inner cylinder,

$$\vec{T}_i = \vec{s} \times Q \vec{E}_i = a \hat{s} \times Q \vec{E}_i$$

$$= a Q \left(-\frac{1}{2} a \mu_o n \frac{dI}{dt} \right) \hat{s} \times \hat{\phi}$$

$$= -\frac{1}{2} Q a^2 \mu_o n \frac{dI}{dt} \hat{z}$$

and the change in angular momentum

$$\Delta \vec{L}_i = \frac{1}{2} Q a^2 \mu_o n I_o \hat{z}$$

Therefore, the angular momentum picked up by the rotating cylinders is

$$\Delta\vec{L}_i + \Delta\vec{L}_o = \frac{1}{2} Q a^2 \mu_0 n I_0 - \frac{1}{2} Q R^2 \mu_0 n I \hat{z}$$

$$= \frac{1}{2} Q (R^2 - a^2) \mu_0 n I$$

which is equal to angular momentum lost to the fields.

Note, how quickly the simplicity of the situation collapses.

Suppose the inner cylinder is removed, the angular momentum of the field disappears in the infinite approximation, but the outer cylinder still rotates.