Conservation

Consider some of the conservation laws we work with in physics.

Conservation of Mass (ρ = mass density)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Conservation of Charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

Conservation of Energy

$$\sum U_{before} = \sum U_{after}$$

Conservation of Momentum

$$\sum \vec{P}_{before} = \sum \vec{P}_{after}$$

Note, how much weaker the statements of conservation of energy and momentum are. For charge and mass, we can see how the quantity is conserved. If we separate the universe into a system and the environment separated by a surface S surrounding a volume V, then a change in the charge of the system results from a flow of charge across the boundary.

$$\int_V \frac{\partial \rho}{\partial t}\, d\tau + \int_V \nabla \cdot \vec{J}\, d\tau = 0$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$\frac{d}{dt}\underbrace{\int_V \rho\, d\tau}_{\text{total charge of system}} + \underbrace{\int_S \vec{J} \cdot d\vec{a}}_{\text{flow of charge out of system}} = 0$$

For momentum and energy, we have no idea how they are conserved. For all we know, energy pixies teleport energy between the system and the environment.

We will rectify this lack of precision by writing a continuity equation for electromagnetic energy.

Consider a system of charged particles isolated from the environment. The moving particles create electric and magnetic fields. The fields do work on the other charges changing their kinetic and potential energy.

Consider the work done by the EM fields on the charged particles per unit time, W

$$\frac{dW}{dt} = \frac{d}{dt}\int_V \vec{F}_{em}\cdot d\vec{l} = \cancel{\int_V \vec{F}_{em}\cdot\vec{v}\,d\tau}$$

electromagnetic force

$$\vec{F}_{em} = q\vec{E} + q\vec{v}\times\vec{B}$$

The force per unit volume is

$$\vec{f}_{em} = \rho\vec{E} + \rho\vec{v}\times\vec{B}$$

$$\frac{dW}{dt} = \int_V (\vec{f}_{em} \cdot \vec{v})\, d\tau \qquad \frac{d\vec{l}}{dt} = \vec{v}$$

$$\vec{v} \cdot \vec{f}_{em} = \rho \vec{v} \cdot \vec{E} \qquad (\vec{v} \cdot (\vec{v} \times \vec{B}) = 0$$

$$= \vec{J} \cdot \vec{E}$$

$$\frac{dW}{dt} = \int_V \vec{E} \cdot \vec{J}\, d\tau$$

$\Longrightarrow$ This is simply the Joule heating term we encountered before.

Eliminate $\vec{J}$ using Ampere's Law

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \vec{E} \cdot (\nabla \times \vec{B}) - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

<u>Work on the $\vec{E} \cdot (\nabla \times \vec{B})$ term</u>

Using a product rule

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\cancel{\vec{E} \rightarrow \vec{B} \text{ and } \vec{B} \rightarrow \vec{A}}$$

$$\cancel{\nabla \cdot (\vec{E} \times \vec{B}) = \vec{E} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{E})}$$

So $\quad \nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{B})$

$$\Rightarrow \vec{E} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{B})$$

Use Faraday's Law on the second term

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \vec{E} \cdot (\nabla \times \vec{B}) = -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \nabla \cdot (\vec{E} \times \vec{B})$$

So,

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0}\left(-\vec{B} \cdot \frac{\partial \vec{B}}{\partial t}\right) - \frac{1}{\mu_0} \nabla \cdot (\vec{E} \times \vec{B})$$

$$- \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial B^2}{\partial t} \qquad (B^2 = \vec{B} \cdot \vec{B})$$

$$\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial E^2}{\partial t} \qquad E^2 = \vec{E} \cdot \vec{E}$$

$$\vec{E}\cdot\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})$$

and

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0}\right)d\tau$$

$$- \int \frac{\nabla\cdot(\vec{E}\times\vec{B})}{\mu_0}\,d\tau$$

The first term is the energy stored in the electromagnetic fields

$$U_{em} = \int_V\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0}\right)d\tau$$

Use the divergence thm on the second term

$$\frac{dW}{dt} = -\underbrace{\frac{d}{dt}U_{em}}_{\text{energy stored in fields}} - \underbrace{\oint_S \frac{\vec{E}\times\vec{B}}{\mu_0}\cdot d\vec{a}}_{\text{energy flow out of surface}}$$

Work Energy Thm

Poynting's Thm

The energy flux out of the surface bounding the volume is given by $\vec{S} \equiv \frac{\vec{E} \times \vec{B}}{\mu_0}$ which is called the Poynting vector.

$$\left[|\vec{S}| \right] = \frac{\text{Energy}}{\text{Area} \cdot \text{Time}} = \frac{\text{Power}}{\text{Time}}$$

The work done on the particles increases the mechanical energy U_{mech}. If this can be written as a density, u_{mech} where

$$U_{mech} = \int_V u_{mech} \, d\tau$$

Then the Work-Energy Thm can be written as a continuity eqn for energy

$$\frac{\partial}{\partial t} \left(u_{mech} + u_{em} \right) + \nabla \cdot \vec{S} = 0$$

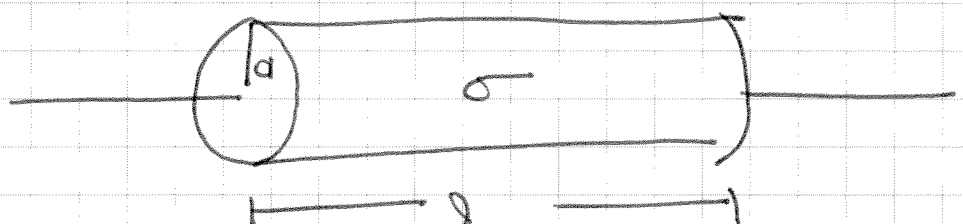
$$u_{em} = \frac{1}{2} \varepsilon_0 E^2 + \frac{B^2}{2\mu_0}$$

Ex If a current I flows uniformly in a resistor of radius a, a current density J is produced

$$J = \frac{I}{\pi a^2}$$

which requires an electric field by Ohm's Law

$$E = \frac{J}{\sigma}$$



Since the field is uniform, the potential difference is $\Delta V = E\ell$

$$= \frac{I\ell}{\pi a^2 \sigma} = IR$$

The energy converted from EM to mechanical (heat) is

$$P = I\Delta V = \frac{I^2 \ell}{\pi a^2 \sigma}$$

We could calculate this directly from Joule heating

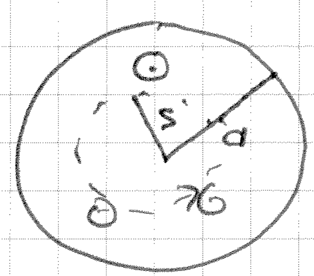
$$P = \int \vec{E} \cdot \vec{J} \, d\tau = \frac{1}{\sigma} \int J^2 \, d\tau$$

$$= \frac{1}{\sigma} \left(\frac{I}{\pi a^2} \right)^2 \pi a^2 l$$

$$= \frac{I^2}{\sigma \pi a^2} l$$

or from the ~~Work~~ Poynting's Thm

<u>Magnetic Field in Resistor</u>



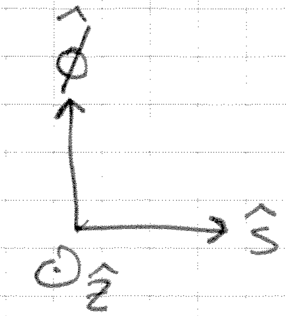
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \pi s^2 J = \mu_0 I \frac{\pi s^2}{\pi a^2}$$

$$= 2\pi s B$$

$$\vec{B} = \frac{\mu_0 I s}{2\pi a^2} \hat{\phi}$$

$$\vec{E} = \frac{I}{\pi a^2 \sigma} \hat{z}$$

Poynting Vectors



$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$$

$$= \frac{1}{\mu_0}\left(\frac{I}{\pi a^2 \sigma}\right)\hat{z} \times \left(\frac{\mu_0 I s}{2\pi a^2}\right)\hat{\phi}$$ Evaluate at $s = a$

$$= \frac{I^2}{2\pi^2 a^3 \sigma}\left(-\hat{s}\right)$$ $\hat{z} \times \hat{\phi} = -\hat{s}$

The energy flux out of the ends of the cylinder is zero since $\vec{S} \perp \hat{z}$.

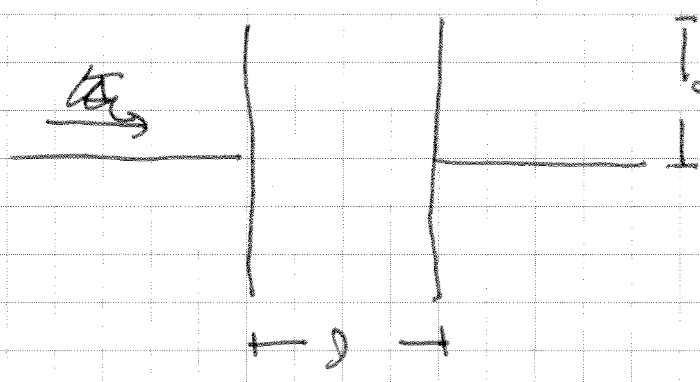
The energy flux out of the curved sides is

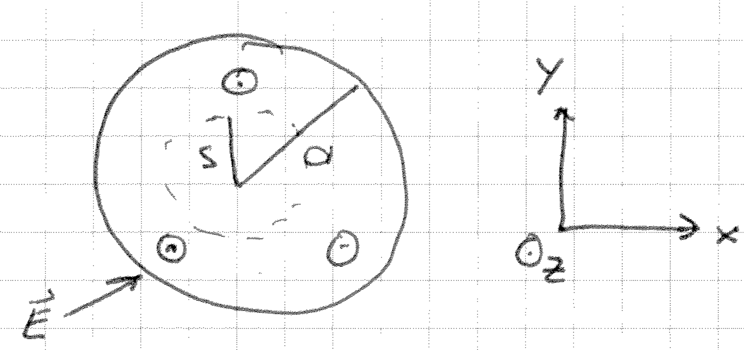
$$\int_S \vec{S} \cdot d\vec{a} = -2\pi a \ell |\vec{S}|$$

$$= \frac{-I^2 \ell}{\pi a^2 \sigma} = I \Delta V$$

So a flow of electromagnetic energy through the curved side of the resistor is converted into thermal (mechanical) energy in the resistor.

<u>Ex</u> Return to our discharging capacitor with electric field between the plates of $\vec{E} = E_0 e^{-t/\tau}\hat{z}$ where the plates have radius a.





We calculated the magnetic field as

$$\vec{B} = -\frac{\mu_0 \varepsilon_0 E_0 s}{2\tau} e^{-t/\tau}\hat{\phi}$$

The Poynting Vector for this system is

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)$$

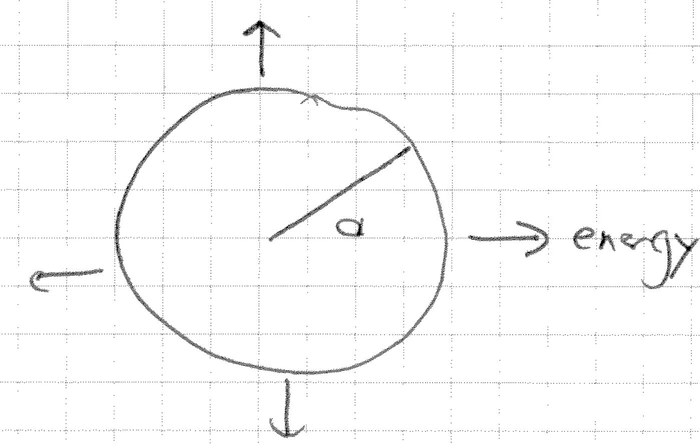
$$= \frac{1}{\mu_0}\left(E_0 e^{-t/\tau}\hat{z}\right)\times\left(\frac{-\mu_0 \varepsilon_0 E_0 a}{2\tau} e^{-t/\tau}\hat{\phi}\right)$$

where $s = a$, the outer edge of the capacitor.

$$\vec{S} = \frac{-\varepsilon_0 E_0^2 a}{2\tau} e^{-2t/\tau}\,\hat{z}\times\hat{\phi}$$

$$= \frac{\varepsilon_0 E_0^2 a}{2\tau} e^{-2t/\tau}\,\hat{s} \qquad \left(-\hat{s} = \hat{z}\times\hat{\phi}\right)$$

$\Longrightarrow$ There is a flow of energy out of the edges of the capacitor.



The total energy flowing out of the capacitor per unit time is

$$P = \frac{\text{Energy}}{\text{Time}} = SA = S \cdot 2\pi a l$$

$$= \frac{\varepsilon_0 E_0^2 a}{2\tau} \cdot 2\pi a l \, e^{-2t/\tau}$$

$$= \frac{\varepsilon_0 E_0^2 \pi a^2 l}{\tau} e^{-2t/\tau}$$

Check this power against the energy stored in the capacitor.

$$U = \frac{1}{2} C (\Delta V)^2$$

$$\frac{dU}{dt} = C \Delta V \frac{d\Delta V}{dt}$$

$$C = \frac{\varepsilon_0 A}{l} = \frac{\varepsilon_0 \pi a^2}{l}$$

$$\Delta V = E l$$

$$\frac{d\Delta V}{dt} = l \frac{dE}{dt} = -\frac{E_0 l}{\tau} e^{-t/\tau}$$

$$\frac{dU}{dt} = C\Delta V \frac{d\Delta V}{dt}$$

$$= \left(\frac{\varepsilon_0 \pi a^2}{l}\right)\left(E_0 l e^{-t/\tau}\right)\left(-\frac{E_0 l}{\tau} e^{-t/\tau}\right)$$

$$= -\frac{\varepsilon_0 \pi a^2 E_0^2 l}{\tau} e^{-2t/\tau}$$

$\Rightarrow$ We could have also checked this against the energy density in the electromagnetic field.