Coulomb's Law

With the Law of Linear Superposition, we could build up any electric field by summing the electric field of a point charge, Q.

The charge density of a point charge at location $\vec{r}\,'$ is

$$\rho(\vec{r}) = Q\,\delta^3(\vec{r}-\vec{r}\,') = Q\,\delta^3(\vec{r}\,'')$$

Notation

We will use $\vec{r}\,'$ for the location of sources, like charges, and $\vec{r}$ for the location of the field.

The displacement vector from the source point to the field point will be written $\vec{r}\,'' = \vec{r} - \vec{r}\,'$.

Griffith's uses $\vec{\mathscr{r}} = \vec{r}\,''$, but no one else does.

If there are multiple sources i, we will write the multiple displacement vectors as

$$\vec{r}_i\,'' = \vec{r} - \vec{r}_i\,'$$

$$\int_V \rho(\vec{r})\, d\tau = \begin{cases} Q & \text{, if } V \text{ contains } \vec{r}\,' \\ 0 & \text{, if not} \end{cases}$$

<u>Solve Maxwell's Eqns</u>

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 = \frac{q}{\varepsilon_0}\,\delta^3(\vec{r}\,'')$$

$$\nabla \times \vec{E} = 0$$

We know $\nabla \cdot \left(\dfrac{\hat{r}''}{r''^2} \right) = 4\pi\,\delta^3(\vec{r}\,'')$

so try $\vec{E} = \dfrac{C\,\hat{r}}{r''^2}$

$$\nabla \cdot \vec{E} = 4\pi C\,\delta^3(\vec{r}\,'') = \frac{q}{\varepsilon_0}\,\delta^3(\vec{r}\,'')$$

$$\Longrightarrow C = \frac{q}{4\pi\varepsilon_0}$$

therfore the electric field of a point charge is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r''^2}\,\hat{r}''$$

If wee let $\vec{r}\,' = 0$, the point charge is at the origin,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

and $\nabla \times \vec{E} = 0$ using the front cover.

<u>Coulomb's Law</u> The electric field at point $\vec{r}$ due to a point charge q at $\vec{r}\,'$ is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r''^2}\hat{r}''$$

Now find the electric potential of a point charge

$$\nabla\left(\frac{1}{r''}\right) = -\frac{\hat{r}''}{r''^2}$$

$$\vec{E} = -\nabla V$$

$$V = \frac{q}{4\pi\varepsilon_0 r''} + C$$

arbitrary constant.

If we choose $\vec{r}_0 = \infty$ and set $V(\vec{r}_0) = 0$

$\Rightarrow C = 0.$

<u>Electric Potential of a Point Charge</u> - The electric potential at the point $\vec{r}$ due to a point charge q at $\vec{r}\,'$ is

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r''} = \frac{kq}{r''}$$

where $k = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9 \frac{Nm^2}{C^2}$

Using Linear Superposition we can build up the field of complicated charge distributions

$$\vec{E}(\vec{r}) = \sum_i \frac{q_i}{4\pi\varepsilon_0 r_i''^2} \hat{r}_i''$$

$$V(\vec{r}) = \sum_i \frac{q_i}{4\pi\varepsilon_0 r_i''}$$

An alternate derivation of the potential of a point charge

$$V(\vec{r}) = - \int_{\vec{r}_0 \to \vec{r}} \vec{E} \cdot d\vec{l}$$

$$= - \int_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r} \cdot (\hat{r}\, dr)$$

$$= \frac{Q}{4\pi\varepsilon_0 r}$$

Using Linear Superposition we can also integrate over continuous charge distribution. Remember we always sum or integrate over sources.

$$\vec{E}(\vec{r}) = \int_V \frac{\rho(\vec{r}\,')\,\hat{r}''\,d\tau'}{4\pi\varepsilon_0 r''^2}$$

volume charges

$$V(\vec{r}) = \int_V \frac{\rho(\vec{r}\,')\,d\tau'}{4\pi\varepsilon_0 r''}$$

$$\vec{E}(\vec{r}) = \int_S \frac{\sigma(\vec{r}\,')\,\hat{r}''\,da'}{4\pi\varepsilon_0 r''^2}$$

surface charges

$$V(\vec{r}) = \int \frac{\sigma(\vec{r}\,')\,da'}{4\pi\varepsilon_0 r''}$$

$$E(\vec{r}) = \int_C \frac{\lambda(\vec{r}\,')\,\hat{r}''\,d\ell'}{4\pi\varepsilon_0 r''^2}$$

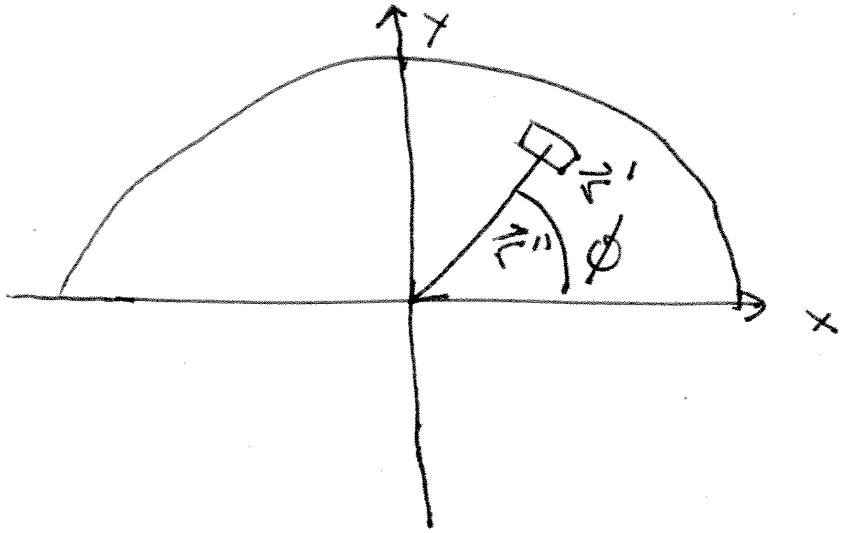
linear charges

$$V(\vec{r}) = \int_C \frac{\lambda(\vec{r}\,')\,d\ell'}{4\pi\varepsilon_0 r''}$$

Note how much more simple the potential formulas are.

<u>Ex</u> Compute $\vec{E}$ at the origin of a uniformly charged half-circle of radius R s.t.

σ = constant for $y>0$ and $\sigma = 0$ for $y<0$.



<u>Sln</u>

The field point is at $\vec{r} = (0,0,0)$.

<u>A</u> source point is at $\vec{r}' = s'\hat{s}' = \vec{s}'$

The displacement vector from the source point to the field point is

$$\vec{r}'' = \vec{r} - \vec{r}' = -s'\hat{s}'$$

Apply Coulomb's Law and the Law of Linear Superposition

$$\vec{E}(\vec{r}) = \int_S \frac{\sigma(\vec{r}')\, da'}{4\pi\varepsilon_0 r''^2}\, \hat{r}''$$

$$\vec{E} = -\int_S \frac{\sigma \, da'}{4\pi\varepsilon_0 s'^2} \hat{s}'$$

where $\sigma(\vec{r}\,') = \sigma$ is constant

$r'' = s'$ $\qquad \hat{r}'' = -\hat{s}'$

The area element in cylindrical coordinates is

$$da' = (ds')(s' d\phi')$$

$$\vec{E}(\vec{r}) = \frac{-\sigma}{4\pi\varepsilon_0} \int_0^R ds' \int_0^\pi d\phi' \frac{s' \hat{s}'}{s'^2}$$

$$\cancel{= \frac{-\sigma}{4\pi\varepsilon_0} \left[\int_0^\pi d\phi' \right] \left[\int_0^R ds' \frac{\hat{s}'}{s'} \right]}$$

$\Rightarrow$ <u>Problem</u> $\hat{s}'$ changes direction as we integrate ϕ'

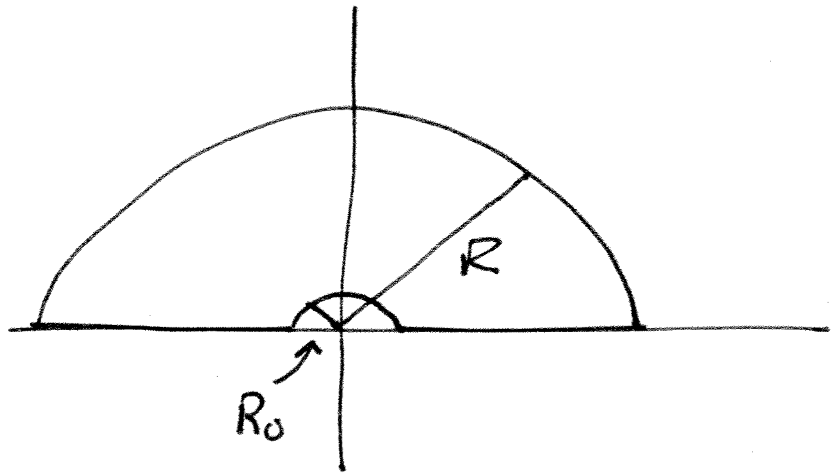
<u>Solution</u> Express it in terms of $\hat{x}', \hat{y}', \hat{z}'$.

Note $\hat{x}' = \hat{x} = (1,0,0)$ etc.

From Griffiths cover

$$\hat{s}' = \cos\phi' \, \hat{x}' + \sin\phi' \, \hat{y}'$$

<u>Problem</u> The integral is singular. Cut a small hole around the field point



We may still be able to take the limit $R_0 \to 0$ when we are done (or maybe not).

$$\vec{E}(\vec{r}) = \frac{-\sigma}{4\pi\epsilon_0}\hat{x}\int_0^{\pi} d\phi' \int_{R_0}^{R} \frac{ds'}{s'}\cos\phi'$$

$$\frac{-\sigma}{4\pi\epsilon_0}\hat{y}\int_0^{\pi} d\phi' \int_{R_0}^{R} \frac{ds'}{s'}\sin\phi'$$

$$\int_0^{\pi}\cos\phi' \, d\phi' = 0 \qquad \int_0^{\pi}\sin\phi' \, d\phi' = 2$$

$$\Downarrow$$

the $\hat{x}$ term is zero

$$\vec{E}(\vec{r}) = \frac{-2\sigma}{4\pi\varepsilon_0}\hat{y}\int_{R_0}^{R}\frac{ds'}{s'}$$

$$= \frac{-\sigma}{2\pi\varepsilon_0}\hat{y}\ln\left(\frac{R}{R_0}\right)$$

so the singularity is real and we cannot let $R_0 \to 0$.

* Note, the field points in the direction it had to.

<u>Ex</u> Compute the potential at the origin of the same system.

$$V(\vec{r}) = \int_s \frac{\sigma(\vec{r}')\,da'}{4\pi\varepsilon_0 \mathscr{r}} \qquad da' = ds's'd\phi' \quad \mathscr{r} = s'$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\underbrace{\int_0^{\pi} d\phi'}_{\pi}\underbrace{\int_{R_0}^{R}\frac{s'ds'}{s'}}_{R-R_0}$$

$$= \frac{\sigma}{4\varepsilon_0}(R-R_0)$$

Note, we cannot use $\vec{E} = -\nabla V$ to calculate the field here because we only calculated the potential at one point.