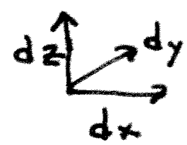
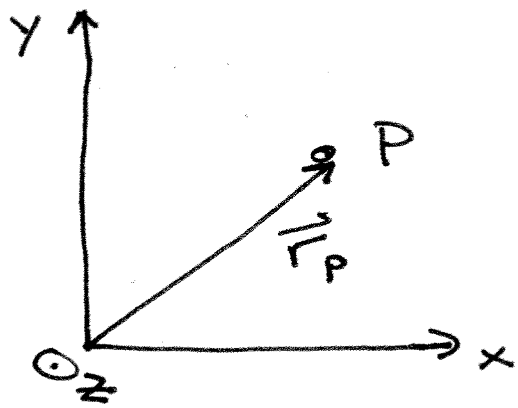
Curvilinear Coordinates

Cartesian $\hat{x} \times \hat{y} = \hat{z}$ right-handed triple

Volume element $d\tau = dx\, dy\, dz$



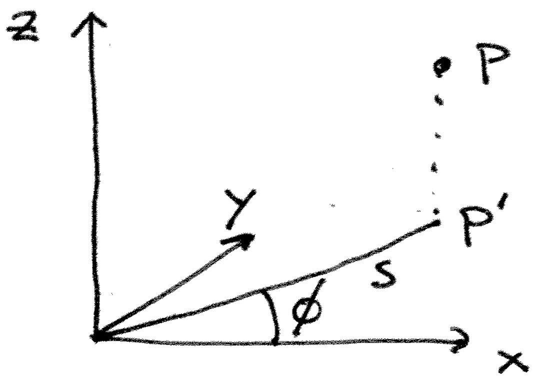


$$\vec{r}_P = x\hat{x} + y\hat{y} + z\hat{z} = (x, y, z)$$

Path element $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

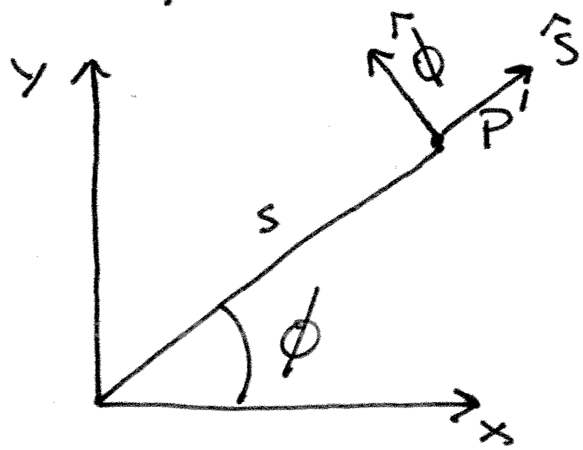
Cylindrical Coordinates

$\hat{s} \times \hat{\phi} = \hat{z}$ right-handed triple



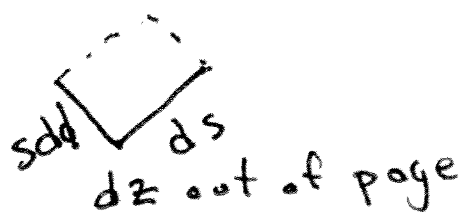
P' = projection of P on x-y plane

In x-y plane,



Volume Element

$$d\tau = (ds)(s\,d\phi)\,dz$$



Path Element

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

$\vec{r}_P$ is a vector from the origin to P. It can be expressed in any coordinate system.

$$\vec{r}_P = x\hat{x} + y\hat{y} + z\hat{z} \equiv f_x\hat{x} + f_y\hat{y} + f_z\hat{z}$$

$$= f_s\hat{s} + f_\phi\hat{\phi} + f_z\hat{z}$$

$\hat{x}, \hat{y}, \hat{z}$ orthonormal , $\hat{s}, \hat{\phi}, \hat{z}$ orthonormal

$$f_s = \hat{s}\cdot\vec{r}_P = f_x\,\hat{s}\cdot\hat{x} + f_y\,\hat{s}\cdot\hat{y} + f_z\,\hat{s}\cdot\hat{z}$$

Now we need the dot products.

Transformation Equations

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

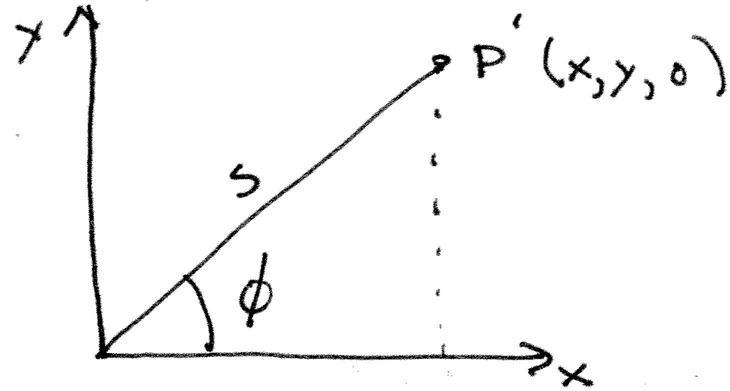
$$\hat{z} = \hat{z}$$

$$\hat{s}\cdot\hat{x} = \cos\phi \qquad \hat{s}\cdot\hat{y} = \sin\phi$$

$$\hat{s}\cdot\hat{z} = 0$$

So, $f_s = f_x \cos\phi + f_y \sin\phi$

For our $\vec{r}_P$, $f_x = x$, $f_y = y$



$$s = \sqrt{x^2+y^2} \qquad \cos\phi = \frac{x}{s} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_s = x\cos\phi + y\sin\phi$$

$$= \frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = s$$

A similar process will determine f_ϕ. $f_z = f_z$

cartesian cylindrical

Could also use transformation equ

$$x = s\cos\phi \qquad y = s\sin\phi$$

$$f_s = s\cos^2\phi + s\sin^2\phi = s$$

Vector Operators in Cylindrical Coordinates

$$\nabla \neq \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial z} \right) \quad \text{because}$$

$\hat{s}, \hat{\phi}$ change with position. $\Rightarrow$ The unit vectors contribute to the gradient.

$\Rightarrow$ Look up the correct form in Griffiths cover.

__Ex__ Compute $\nabla \cdot (s\hat{\phi})$

$$\vec{A} = (A_s, A_\phi, A_z) = A_s\hat{s} + A_\phi\hat{\phi} + A_z\hat{z}$$

$$= (0, s, 0)$$

$$A_s = 0, \quad A_\phi = s, \quad A_z = 0$$

__From front cover__

$$\nabla \cdot \vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\nabla \cdot (s\hat{\phi}) = \frac{1}{s}\frac{\partial}{\partial s}(s \cdot 0) + \frac{1}{s}\frac{\partial s}{\partial \phi} + \frac{\partial 0}{\partial z}$$

$$= 0$$

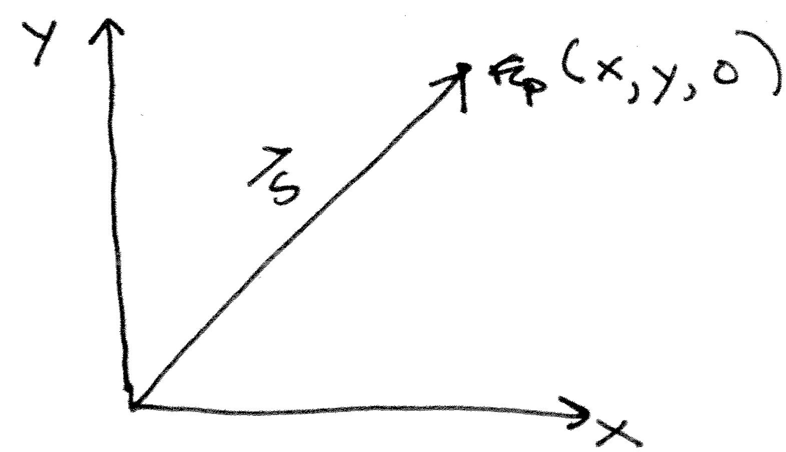
Don't differentiate twice. The derivative formulas take care of the change in $\hat{\phi}, \hat{r}, \hat{s}, \hat{\theta}$.

$\Rightarrow$ When in doubt, use cartesian.

Ex Is $\nabla \cdot \hat{s} = 0$? (it would be if $\hat{s} \to \hat{x}$)

$$\hat{s} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0 \right)$$

since $\vec{s} = (x, y, 0)$.

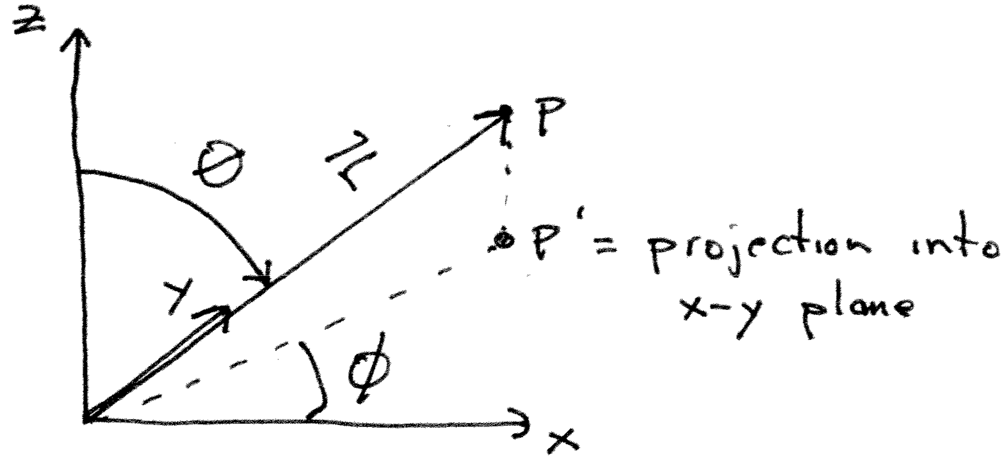


$$\nabla \cdot \hat{s} = \left(\frac{\partial}{\partial x} \frac{x}{\sqrt{x^2+y^2}} + \frac{\partial}{\partial y} \frac{y}{\sqrt{x^2+y^2}} + 0 \right)$$

$$\neq 0$$

Spherical Coordinates

$\hat{r} \times \hat{\theta} = \hat{\phi}$ right-handed triple



* Length O - P', $r \sin\theta$
* If $\vec{r}_P = (x, y, z)$

$$r = |\vec{r}_P| = \sqrt{x^2+y^2+z^2}$$

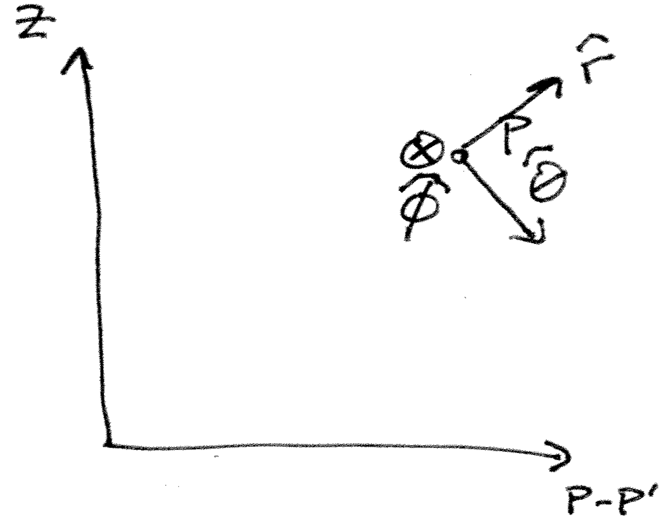
Line Element

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

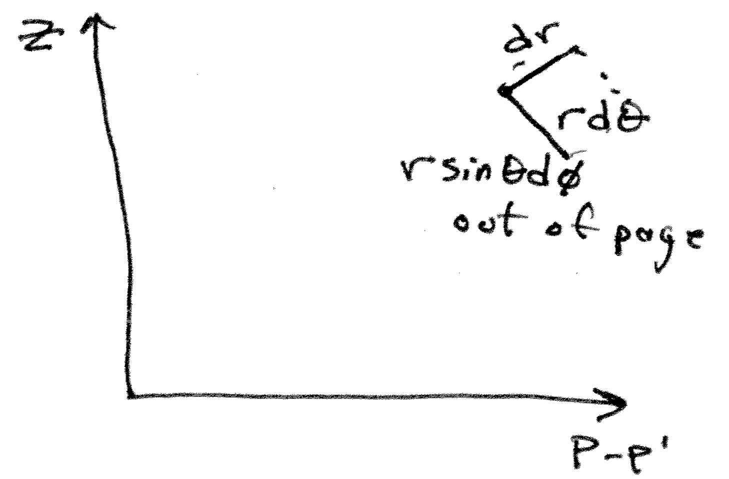
Volume Element

$$d\tau = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$$

In the plane containing P, P'



<u>Volume</u>



Ex Suppore $\vec{r}_P = (x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$

$= f_r \hat{r} + f_\theta \hat{\theta} + f_\phi \hat{\phi}$

what is f_θ?

$$f_\theta = \hat{\theta} \cdot \vec{r}_P = x(\hat{x} \cdot \hat{\theta}) + y(\hat{y} \cdot \hat{\theta}) + z(\hat{z} \cdot \hat{\theta})$$

Transformation Equation (Back cover)

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{x} \cdot \hat{\theta} = \cos\theta\cos\phi$$

$$\hat{y} \cdot \hat{\theta} = \cos\theta\sin\phi$$

$$\hat{z} \cdot \hat{\theta} = -\sin\theta$$

$$f_\theta = x\cos\theta\cos\phi + y\cos\theta\sin\phi - z\sin\theta$$

Not quite complete   Want x, y, z in terms of r, θ, ϕ

$$z = r\cos\theta$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$f_\theta = r\cos\theta\sin\theta\cos^2\phi + r\cos\theta\sin\theta\sin^2\phi - r\cos\theta\sin\theta$$

$$= r\cos\theta\sin\theta - r\cos\theta\sin\theta = 0$$

Does that make sense?

Sure $\vec{r}_P = r\hat{r}$

TOPS 35502

Ex Find $\nabla^2 r$

Front Cover

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$\nabla^2 r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial r}{\partial r}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial r}{\partial\theta}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial^2 r}{\partial\phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\right) + 0 + 0$$

$$= \frac{1}{r^2}\cdot 2r = \frac{2}{r}$$