Displacement

The bound charge that results from polarization is just charge and needs to be treated the same way as any other net charge in the system.

$$\nabla \cdot \vec{E} = \frac{\rho_b}{\varepsilon_0}$$

$$\nabla \times \vec{E} = 0$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_b(\vec{r}\,')\,\hat{r}''\,d\tau'}{(r'')^2}$$

If there is additional charge floating around not produced by polarization, it just gets added in

$$\nabla \cdot \vec{E} = \frac{\rho_b}{\varepsilon_0} + \frac{\rho_f}{\varepsilon_0}$$

where ρ_f is the charge NOT resulting from polarization. ρ_f is called the <u>free charge</u>. as opposed to the <u>bound charge</u> ρ_b.

Let's rearrange

$$\varepsilon_0 \nabla \cdot \vec{E} = \rho_f + \rho_b$$

$$\rho_b = -\nabla \cdot \vec{P}$$

$$\varepsilon_0 \nabla \cdot \vec{E} = \rho_f - \nabla \cdot \vec{P}$$

$$\varepsilon_0 \nabla \cdot \vec{E} + \nabla \cdot \vec{P} = \rho_f$$

$$\nabla \cdot \left(\varepsilon_0 \vec{E} + \vec{P} \right) = \rho_f$$

Dfn Displacement Field ($\vec{D}$)

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\nabla \cdot \vec{D} = \rho_f$$

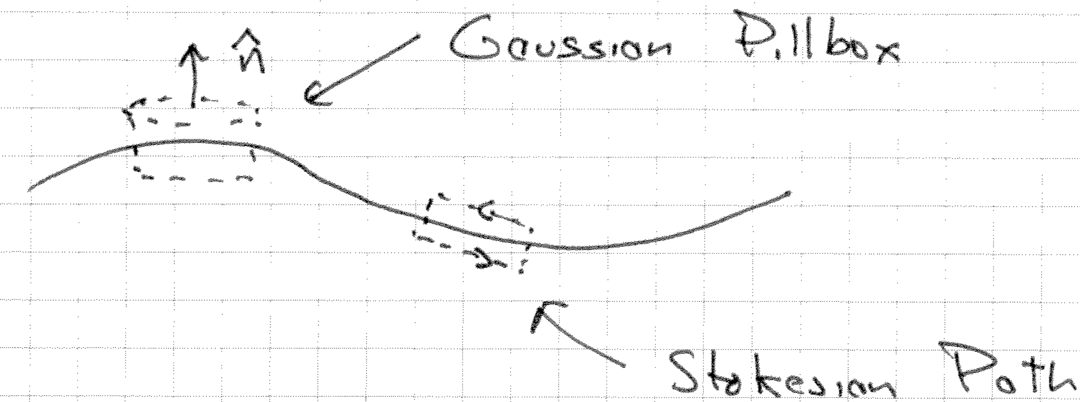
Integral Form

$$\int_V \nabla \cdot \vec{D} \, d\tau = \int_V \rho_f \, d\tau = \oint_S \vec{D} \cdot \hat{n} \, da$$

$$Q_{f,enc} \equiv \int_V \rho_f \, d\tau$$

$$\Phi_d \equiv \oint_S \vec{D} \cdot d\vec{a} = Q_{f\,enc} \qquad d\vec{a} = \hat{n} \, da$$

Electrostatic Boundary Condition



Pillbox

$$\Phi_d = \vec{D}_{top}\cdot\hat{n}\,A - \vec{D}_{bottom}\cdot\hat{n}\,A$$
$$= \sigma_f A$$

$$D^{\perp}_{top} - D^{\perp}_{bottom} = \sigma_f$$

$\perp$ component to surface.

Stokesian Path

$$\oint_c \vec{E}\cdot d\vec{l} = \int_s \nabla\times\vec{E}\cdot d\vec{a} = 0$$

$$= \int_s \nabla\times\left(\frac{\vec{D}}{\varepsilon_0} - \frac{\vec{P}}{\varepsilon_0}\right)\cdot d\vec{a} = 0$$

$$\varepsilon_0\vec{E} + \vec{P} = \vec{D} \implies \vec{E} = \frac{\vec{D}}{\varepsilon_0} - \frac{\vec{P}}{\varepsilon_0}$$

$$\int_S \nabla\times\vec{D}\cdot d\vec{a} = \int \nabla\times\vec{P}\cdot d\vec{a}$$

$$\oint_C \vec{D}\cdot d\vec{l} = \int_C \vec{P}\cdot d\vec{l}$$

<u>For the Stokesian Path</u>

$$D^{\parallel}_{top} - D^{\parallel}_{bottom} = P^{\parallel}_{top} - P^{\parallel}_{bottom}$$

$\parallel \rightarrow$ component of field $\parallel$ to surface.

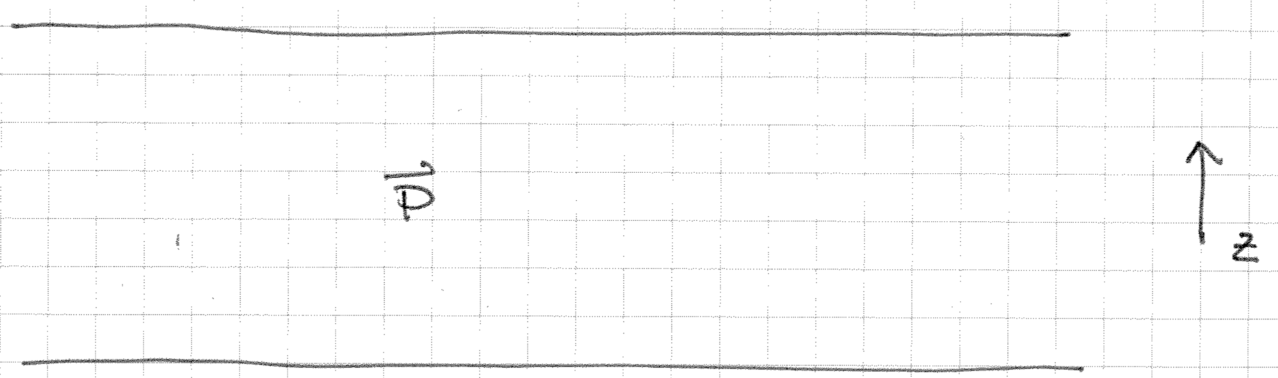
<u>Be careful</u>

$$\nabla\times\vec{D} = \varepsilon_0\,\nabla\times\vec{E} + \nabla\times\vec{P} = \nabla\times\vec{P} \neq 0$$

so

$$\vec{D} \neq \int \frac{k\,\rho_f\,d\tau'}{r''^2}\,\hat{r}_{''}$$

Ex Back to our slab with uniform polarization $\vec{P} = P\hat{z}$



There is no free charge anywhere,

$$\Phi_d = \oint_s \vec{D} \cdot d\vec{a}\,' = Q_{fenc}$$

$$= 0$$

$$\Rightarrow \quad \vec{D} = 0$$

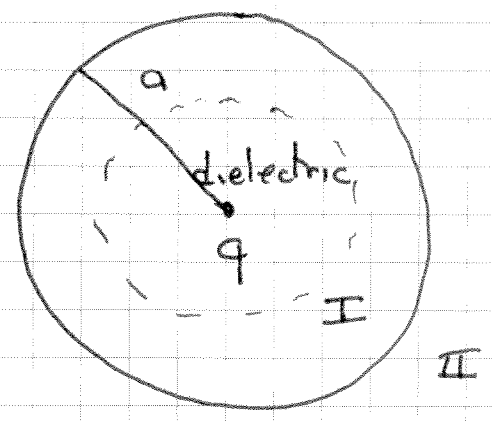
Inside $\quad \vec{D} = 0 = \varepsilon_0 \vec{E} + \vec{P}$

$$\vec{E} = -\frac{\vec{P}}{\varepsilon_0}$$

Outside $\quad \vec{P} = 0$

$$\vec{D} = 0 = \varepsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{E} = 0$$

Ex Point Charge in Dielectric Sphere



Sln

Displacement Flux

$$\Phi_d = \int_S \vec{D}\cdot d\vec{a} = Q_{f,enc}$$

Region I $(r<a)$

$$\Phi_d = 4\pi r^2 D = q$$

$$\vec{D} = \frac{q}{4\pi r^2}\hat{r}$$

We cannot calculate $\vec{E}$ because we don't know $\vec{P}$.

Region II $(r > a)$ $\qquad Q_{f,enc} = q$

$$\vec{D} = \frac{q}{4\pi r^2}\hat{r}$$

but $\vec{P} = 0$ so $\vec{D} = \varepsilon_0 \vec{E}$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Ok, so how can we get $\vec{P}$?

In general if we place a block of dielectric in a field, the polarization produced is complicated because we change the atomic energy environment.

For systems with a linear response

$$\vec{P} = \varepsilon_0 \overleftrightarrow{\chi}_e \vec{E}$$

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

susceptibility tensor

Since most applied electric fields are small compared to atomic fields, many systems are well approximated by the linear response.

If the dielectric is also <u>isotropic</u> with the same response in all directions

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \chi_e & 0 & 0 \\ 0 & \chi_e & 0 \\ 0 & 0 & \chi_e \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_y \end{pmatrix}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

The material constant χ_e is the electric <u>susceptibility</u>.

For these linear isotropic materials,

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E}$$

$$= \varepsilon_0 (1+\chi_e) \vec{E}$$

Dfn Permittivity (ε)

$$\varepsilon = \varepsilon_0 (1 + \chi_e)$$

Dfn Relative Permittivity (ε_r)

$$\varepsilon_r = 1 + \chi_e$$

Dfn Dielectric Constant (κ)

$$\kappa = 1 + \chi_e = \varepsilon_r$$

$$\Rightarrow \quad \vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E} = \kappa \varepsilon_0 \vec{E}$$

~~for~~ linear, isotropic dielectrics.

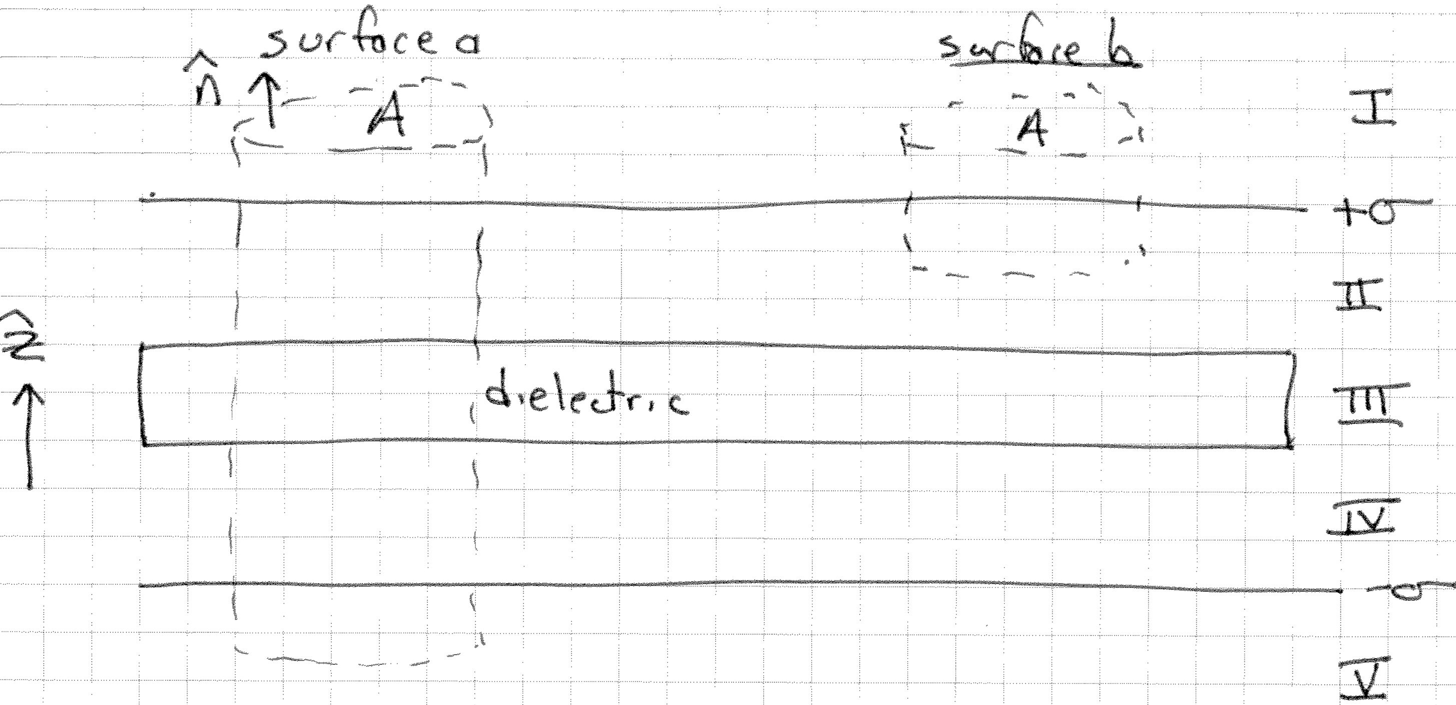
Return to our point charge q in a dielectric sphere with dielectric constant $\kappa = \varepsilon_r$

Inside Sphere ($r < a$)

$$\vec{D} = \frac{q}{4\pi r^2} \hat{r} \qquad \text{as before}$$

$$\vec{E} = \frac{\vec{D}}{\kappa \varepsilon_0} = \frac{q}{4\pi \kappa \varepsilon_0 r^2} \hat{r}$$

Ex Dielectric between two equal, but opposite charged planes $\pm\sigma$



Sln Surface a $Q_{f,enc} = 0 = (\sigma + (-\sigma))A$

Outer fields equal, but opposite by reflection symmetry

$$E_V = -E_I \implies \vec{D}_V = -\vec{D}_I$$

$$\Phi_d = \vec{D}_V \cdot \hat{n}A - \vec{D}_I \cdot \hat{n}A = Q_{fenc} = 0$$

$$(\hat{n} = \hat{z})$$

$$\implies \vec{D}_I = \vec{D}_V = 0$$

Surface b $\qquad Q_{fenc} = \sigma A$

$$\Phi_d = \vec{D}_{I} \cdot \hat{n} A - \vec{D}_{II} \cdot \hat{n} A = Q_{fenc} = \sigma A$$

$$\vec{D}_{II} = D_{II} \hat{z} \qquad \hat{n} = \hat{z}$$

$$-D_{II} A = \sigma A$$

$$\vec{D}_{II} = -\sigma \hat{z}$$

Since there is no free charge between the plates,

$$\vec{D}_{II} = \vec{D}_{III} = \vec{D}_{IV}$$

Region II $\qquad \vec{P}_{II} = 0 \qquad \vec{D}_{II} = \varepsilon_0 \vec{E}_{II}$

$$\vec{E}_{II} = \frac{-\sigma}{\varepsilon_0} \hat{z}$$

Region IV $\qquad \vec{P}_{IV} = 0 \qquad \vec{D}_{IV} = \varepsilon_0 \vec{E}_{IV}$

$$\vec{E}_{IV} = \frac{-\sigma}{\varepsilon_0} \hat{z}$$

Region III (in dielectric)

$$\vec{D}_{III} = \varepsilon_r \varepsilon_0 \vec{E}_{III}$$

$$\vec{E}_{III} = \frac{-\sigma}{\varepsilon_r \varepsilon_0} \hat{z} = \frac{-\sigma}{\kappa \varepsilon_0} \hat{z}$$

Polarization $\quad \vec{P}_{III} = \varepsilon_0 \chi_e \vec{E}_{III}$

$$= \frac{-\sigma \chi_e}{\varepsilon_r} \hat{z}$$

$$= -\sigma \left(\frac{\chi_e}{1+\chi_e} \right) \hat{z}$$

Charge Densities

$$\rho_b = -\nabla \cdot \vec{P} = 0 \quad \text{everywhere}$$

$$\sigma_{b,top} = \vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{z}$$

$$= \frac{-\chi_e}{1+\chi_e} \sigma \qquad \text{top of dielectric}$$

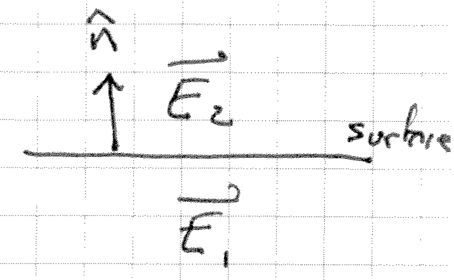
$$\sigma_{b,bottom} = \vec{P} \cdot \hat{n} = \vec{P} \cdot (-\hat{z}) = \frac{\chi_e}{1+\chi_e} \sigma$$

Electrostatic Boundary Conditions

Without Dielectrics

$$\vec{E}_2 \cdot \hat{n} - \vec{E}_1 \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}$$

$$\vec{E}_2 \cdot \hat{t} = \vec{E}_1 \cdot \hat{t}$$



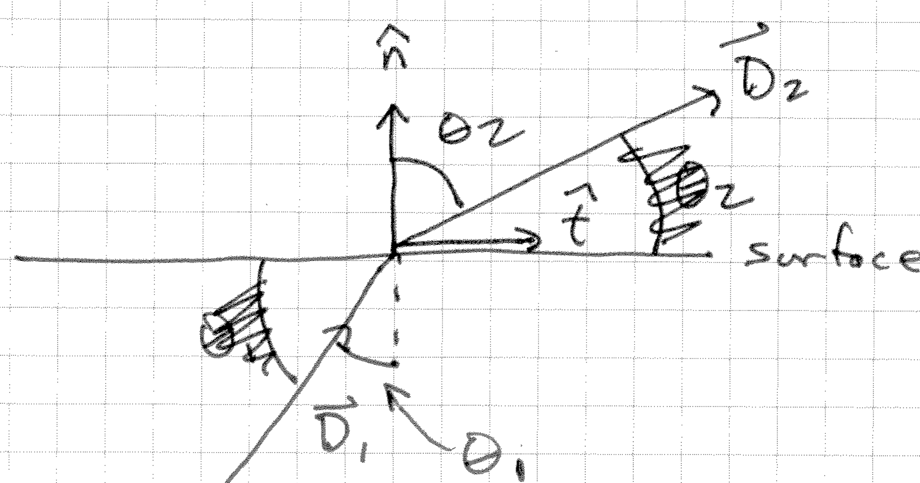
$\hat{n}$ surface normal, $\hat{t}$ tangent to surface

With Dielectrics

$$\vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = \sigma_f$$

$$\vec{E}_2 \cdot \hat{t} = \vec{E}_1 \cdot \hat{t}$$

Suppose $\sigma_f = 0$ at dielectric surface



If $\sigma_f = 0$

$$\vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = 0$$

$$\Downarrow$$

$$D_2 \cos\theta_2 = D_1 \cos\theta_1$$

$$\vec{E}_2 \cdot \hat{t} = \vec{E}_1 \cdot \hat{t} \implies E_2 \sin\theta_2 = E_1 \sin\theta_1$$

$$\frac{E_2}{D_2} \tan\theta_2 = \frac{E_1}{D_1} \tan\theta_1$$

If linear, $D_1 = \varepsilon_r \varepsilon_0 E_1$

$$\frac{1}{\varepsilon_{r2}} \tan\theta_2 = \frac{1}{\varepsilon_{r1}} \tan\theta_1$$

$\implies$ $\vec{D}$ bends at a dielectric interface.

Final Points about Dielectrics

1) In a linear dielectric, since $\vec{D} = K\varepsilon_0 \vec{E}$

$\nabla \times \vec{D} = 0$, but it may not be zero at the interfaces.

If, however, there are <u>no</u> interfaces then

$$\nabla \cdot \vec{D} = \rho_f \quad \text{and} \quad \nabla \times \vec{D} = 0$$

These are the same equations we would have without the dielectric, so we could compute the field $\vec{E}_0$ ignoring the dielectric. The field with the dielectric $\vec{E}_K$ can be found by

$$\vec{D}_0 = \varepsilon_0 \vec{E}_0 \quad \text{(no dielectric)}$$

$$= \vec{D}_K \quad (\vec{D} \text{ with dielectric})$$

$$= \varepsilon_r \varepsilon_0 \vec{E}_K \quad \text{(Field with dielectric)}$$

$$\Rightarrow \vec{E}_K = \frac{\vec{E}_0}{\varepsilon_r} = \frac{\vec{E}_0}{K}$$

$\Rightarrow$ The field is reduced by K.

2) The $\vec{D}$ field has nothing to do with potential

$$V = -\int \vec{E} \cdot d\vec{l}$$

3) For a linear dielectric, the free and bound charge are related.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

$$\nabla \cdot \vec{P} = -\rho_b = \varepsilon_0 \chi_e \nabla \cdot \vec{E}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_r \varepsilon_0}$$

$$\nabla \cdot \vec{P} = -\rho_b = \varepsilon_0 \chi_e \nabla \cdot \left(\frac{\vec{D}}{\varepsilon_r \varepsilon_0} \right)$$

$$= \frac{\varepsilon_0 \chi_e}{\varepsilon_r \varepsilon_0} \nabla \cdot \vec{D}$$

$$= \frac{\chi_e}{1 + \chi_e} \rho_f$$

$$\rho_b = -\frac{\chi_e}{1 + \chi_e} \rho_f$$

$\Rightarrow$ In regions without net free charge,

$$\rho_b = 0.$$

4) Potential Boundary Conditions

$$\hat{D}_2 \cdot \hat{n} - \hat{D}_1 \cdot \hat{n} = \sigma_f$$

$$\cancel{\varepsilon_2 \vec{E}_2 \cdot \hat{n} - \vec{D}_1 \varepsilon} \qquad \varepsilon_2 = \varepsilon_{r2}\varepsilon_0$$

$$\varepsilon_2 \vec{E}_2 \cdot \hat{n} - \varepsilon_1 \vec{E}_1 \cdot \hat{n} = \sigma_f$$

(A) $$\varepsilon_2 \frac{\partial V_2}{\partial n} - \varepsilon_1 \frac{\partial V_1}{\partial n} = -\sigma_f$$

(B) $V_2 = V_1$ at interface always.